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A SURVEY OF STEADY-STATE POROUS FLOW FREE BOUNDARY PROBLEMS.(U)

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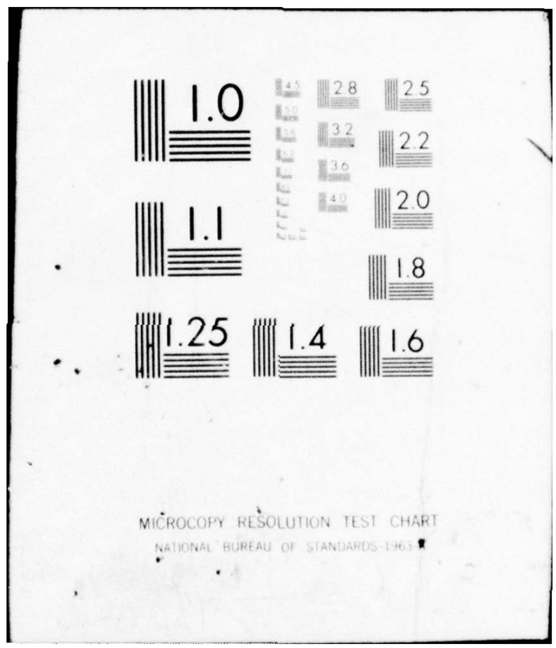
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A SURVEY OF STEADY-STATE POROUS FLOW FREE BOUNDARY PROBLEMS

Colin W. Cryer

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Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

A comprehensive survey of steady-state, porous flow, free boundary (unconfined flow) problems is given. The problems are described and the numerical and analytical approaches which have been used are summarized. Attention is drawn to unsolved problems, and open questions.

AMS (MOS) Subject Classifications: 3502; 35J25; 35R99; 35N99; 76S05

Key Words: Porous flow, Free boundary problems, Unconfined flow problems, Numerical methods, Existence and uniqueness

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A SURVEY OF STEADY-STATE POROUS FLOW FREE BOUNDARY PROBLEMS

Colin W. Cryer

0. Introduction

A free boundary problem (FBP; plural FBPS) is a (steady-state) boundary value problem involving differential equations on domains parts of whose boundaries, the free boundaries (FB; plural FBS) are unknown and must be determined as part of the solution. On such FBS the boundary conditions needed for a fixed boundary value problem (a problem for which the boundaries are known) are supplemented by an additional boundary condition.

FBPS arise in porous flow problems when the porous medium is occupied by two fluids separated by a sharp interface (the FB). The most common FBPS involve water/air interfaces, but water/water vapor, oil/water, oil/gas, and salt water/fresh water interfaces are also often considered.

The following terminology is often used in porous flow FBPS: FBS are called free surfaces or phreatic lines or depression curves or water tables or lines of seepage or floating boundaries or unknown boundaries: FBPS are called unconfined flow problems: the porous medium is called an aquifer.

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Porous flow FBPS have important applications in the fields of soil mechanics, irrigation, ground water hydrology (the control and use of water resources), petroleum engineering, and industrial filtration.

Basic texts on porous flow problems include those of Muskat [1937] and Bear [1972]. More specialized tests include: Polubarinova - Kochina [1962], Harr [1962] (mathematical techniques); Scheidegger [1960, 1963], Childs [1969], Kirkham and Powers [1972] (soil physics); Todd [1959] (ground water hydrology); Carman [1956], R. E. Collins [1961] (gas flow); Luthin [1957], van Schilfgaarde [1974] (drainage engineering); Muskat [1949], Pirson [1958] (petroleum engineering); Hagan, Haise, and Edminster [1967] (irrigation); Remson, Hornberger, and Molz [1971], Oden, Zienkiewicz, Gallagher, and Taylor [1974] (numerical methods). The survey paper of Boersma, Kirkham, Norum, Ziemer, Guitjens, Davidson, and Luthin [1971] draws attention to important recent developments.

Basic journals include:

Advances in Hydroscience,

J. Geophysical Research,

J. Hydraulics Division, Amer. Soc. Civil Engineers,

J. Irrigation and Drainage Division, Amer. Soc. Civil Engineers,

J. Society Petroleum Engineers, Amer. Institute Mechanical Engineers,

J. Soil Science,

Water Resources Research.

Despite the considerable literature on the subject, there is no comprehensive survey of porous flow FBPS. In the present survey we describe and classify all the porous flow FBPS which have, to our knowledge, been considered in the literature. It is hoped that this will reduce duplication of effort and also draw attention to interesting problems which have been neglected.

In porous flow FBPS it is assumed that the flow is <u>saturated</u>, that is, that a particular portion of the porous medium is either saturated with a fluid or contains none, so that there are sharp interfaces between the regions occupied by different fluids. In the context of water/air problems the assumption that the flow is saturated means that a portion of the porous medium is wet or dry but never moist. Porous flow FBPS are thus an approximation to partially saturated flow problems in which the amount of fluid in a particular portion of the porous medium can vary between none and the maximum amount possible. When possible we draw attention to the solutions of partially saturated flow problems which correspond to particular porous flow FBPS, because by comparing such solutions one can check the validity of the assumption of saturated flow.

The present survey is one of a series of surveys on FBPS which we are writing. Other surveys of the series of particular relevance to porous flow FBPS will be those dealing with trial free boundary methods and with variational inequalities.

The remainder of this introduction is organized as follows: in section 0.1 we give a simple example of a porous flow FBP; in section 0.2 we discuss terminology; in section 0.3 we briefly describe some numerical and analytical techniques which are mentioned in the remainder of the survey.

0.1. An example

The problem of seepage through an earth dam (see Figure 1) illustrates how porous flow FBPS arise.

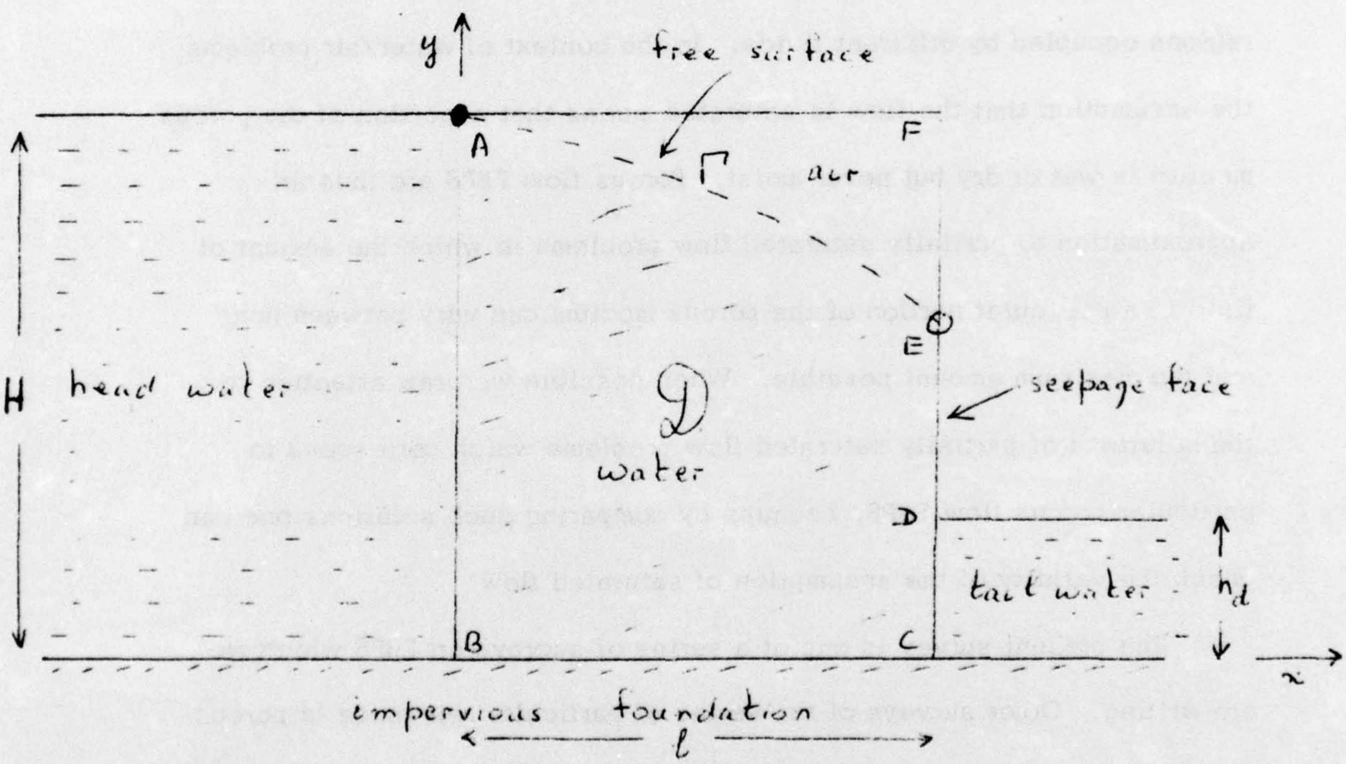


Figure 1: Seepage through a simple rectangular dam

In this problem water from an upstream reservoir (or head water) seeps through a rectangular earth dam to a lower reservoir (or tail water). The water only flows through the region $\mathcal{D}$ and the remainder of the dam remains dry. The air/water interface is denoted by Γ. It is assumed that the dam is so long that the flow is two-dimensional.

If Darcy's law holds (see section 1.1) then the flow is described by a velocity potential ϕ which is related to the hydraulic head h by

$$\phi = -Kh , \tag{1}$$

where the constant K is the hydraulic conductivity. In view of (1) it is immaterial whether the problem is formulated in terms of ϕ or h; here, we usually use h. The hydraulic head satisfies Laplace's equation (see section 1.1):

$$h_{xx} + h_{yy} = 0, \quad \text{in } \mathcal{D} . \tag{2}$$

The boundary conditions are (see section 2):

$$\begin{aligned}
&h = H, \quad \text{on AB (interface with water at rest)} , \\
&\frac{\partial h}{\partial n} = 0, \quad \text{on BC (impervious boundary)} , \\
&h = h_d, \quad \text{on CD (interface with water at rest)} , \\
&h = y, \quad \text{on DE (interface with air)} , \\
&\frac{\partial h}{\partial n} = 0, \quad \text{on EA (streamline)} .
\end{aligned} \tag{3}$$

If Γ were known then equation (2) with boundary conditions (3) would suffice to determine h. Since Γ is not known we have a FBP and an extra condition is required on Γ. This extra condition is:

$$h = y, \quad \text{on EA (interface with air)} . \tag{4}$$

This FBP is discussed further in section 3.1.1.1.

0.2. Terminology

As illustrated by the example in section 0.1, a porous flow FBP has the following structure:

$$\begin{aligned} \mathcal{A}u &= 0, \quad \text{in } \mathcal{D}, \\ \mathcal{B}u &= 0, \quad \text{in } \partial\mathcal{D}, \\ \mathcal{C}u &= 0, \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

where u is the solution and where $\mathcal{A}$ represents the linear or nonlinear elliptic differential equation(s) which must be satisfied by u on the open set $\mathcal{D}$. $\mathcal{B}$ represents the linear or nonlinear boundary conditions which must be satisfied on the boundary $\partial\mathcal{D}$ of $\mathcal{D}$. If $\partial\mathcal{D}$ were completely known then the first two equations of (1) would define u completely. However, part of $\partial\mathcal{D}$, namely the FB Γ, is unknown and $\mathcal{C}$ represents the additional linear or nonlinear boundary condition imposed on Γ.

In (1) it is to be understood that the open set $\mathcal{D}$ may consist of the union of several domains, $\mathcal{D} = \bigcup_{i=1}^{m} \mathcal{D}_i$, in which case the first equation of (1) is equivalent to

$$\mathcal{A}_i u = o \quad \text{in } \mathcal{D}_i, \ 1 \le i \le m .$$

The operators $\mathcal{A}_i$ may or may not be the same.

The domains $\mathcal{D}_i$ will be called <u>free domains</u> (FD; plural, FDS). When it is necessary the number of domains will be indicated by saying that the problem is an <u>m-domain</u> FBP. When $m = 1$ we set $\mathcal{A} \equiv \mathcal{A}_1$, $\mathcal{D} \equiv \mathcal{D}_1$.

Using this terminology, we may describe the example in Section 0.1 as a 1-domain FBP with FD $\mathcal{D}$ and FB Γ.

Most of the porous flow FBPS which have been considered are 1-domain FBPS. A two-domain FBP would arise, for example, in a coastal region in which porous flow of fresh water occurred in a region $\mathcal{D}_1$ and porous flow of salt water occurred in a region $\mathcal{D}_2$, the FB being the salt water/fresh water interface.

It is usually neither possible nor necessary to classify the boundary conditions on the FB into "conventional boundary conditions" and "supplementary boundary conditions", and the division of the boundary conditions on Γ into those represented by $\mathcal{B}$ and that represented by $\mathcal{C}$ is an arbitrary decision.

The FB Γ is defined to be that part of $\partial\mathcal{D}$ whose "shape" is unknown; the remainder of $\partial\mathcal{D}$ will be called the *fixed boundary*. Points at which the FB intersects the fixed boundary will be called *points of detachment*. If the location of a point of detachment is prescribed it is said to be *fixed*; otherwise it is said to be *free*. We illustrate these definitions by considering Figure 1 of section 0.1. The point A is known so that A is a fixed point of detachment. The point E is not known so that E is a free point of detachment. The FB is EA; *although the length of DE is not known its shape is known so that DE is part of the fixed boundary and not part of the FB*.

In figures, the notation of Figure 1 will be used: FBS will be denoted by broken lines, fixed points of detachment by solid circles, and free points of detachment by open circles, while the FD $\mathcal{D}$ will be denoted by hatching.

The FB Γ must usually satisfy certain restraints which we call the <u>auxiliary restraints</u>. The auxiliary restraints arise in several ways:

(i) In some FBPS the boundary operator $\mathcal{C}$ involves an unknown constant. In such FBPS it is often the case that the position or slope of one endpoint of Γ must be prescribed or satisfy an additional condition. FBPS of this type arise, for example, when certain porous flow FBPS are reformulated using the Baiocchi transformation (Baiocchi, Comincioli, Magenes, and Pozzi [1973]).

(ii) It is usually assumed that the solution u satisfies certain smoothness conditions. For example, in the example of section 0.1 it is assumed that the flow velocity at A is finite. It is often a consequence of such assumptions that the slope of Γ at its endpoints is either uniquely determined or limited to a finite number of values; see section 2.2.1.

(iii) Γ must of course satisfy "obvious" restraints such as that it should not intersect itself. While obvious in the physical co-ordinates such restraints become less obvious in transformed co-ordinates.

(iv) In some cases the solution of a FBP is not unique and auxiliary restraints must be imposed in order to eliminate undesirable solutions. For example, Hamel [1938, p. 43] shows that the problem of seepage from a canal has two solutions, one of which is physically unacceptable because the fluid pressure is unbounded.

0.3. Brief summary of some analytical and numerical methods

In the remainder of this survey, reference will be made to some analytical and numerical methods. Here we provide a brief description of these methods. See also Magenes [1972].

The Hodograph Method

There is only one general method for obtaining explicit solutions of FBPS, namely the hodograph method (and variants thereof). The hodograph method relies heavily upon conformal mapping and is essentially restricted to FBPS for which:

(i) The governing differential equation for $h \equiv \phi$ is Laplace's equation in the plane,

$$\phi_{xx} + \phi_{yy} = 0, \quad \text{in } \mathcal{D} .$$

(ii) The fixed boundary, $\partial\mathcal{D}-\Gamma$, is polygonal.

(iii) The boundary conditions are such that there are two distinct conformal maps of $\mathcal{D}$ onto domains of known shape. In particular, let ψ be the complex conjugate of ϕ so that

$f(z) = \phi(x+iy) + i\psi(x+iy)$ is an analytic function of $z = x + iy$. Let $w(z) = df(z)/dz$. Let $\mathcal{D}_f$ and $\mathcal{D}_w$ denote the images of the FD $\mathcal{D}$ in the f and w planes, respectively. If the boundary conditions satisfied by $h \equiv \phi$ are such that $\mathcal{D}_f$ and $\mathcal{D}_w$ are known, then the fact that $df/dz = w$ together with the fact that the conformal mapping of $\mathcal{D}_f$ onto $\mathcal{D}_w$ can be obtained makes it possible to solve the FBP analytically.

The hodograph method has been extensively used with great ingenuity to obtain analytic solutions. Harr [1962], Polubarinova - Kochine [1962] and Bear [1972] describe some of the extraordinary complicated FBPS which have been solved.

Solution in auxiliary planes

In many cases it is convenient to consider a FBP in an auxiliary plane. Thus, with the notation used to describe the hodograph method, one may consider the FBP in the f-plane (or $\phi\psi$-plane) or the w-plane (or hodograph plane) and then solve numerically using finite differences or other methods.

Variational inequalities

An exciting recent discovery due to Baiocchi [1971] has been that many porous flow FBPS can be reformulated as variational inequalities. In the case of the example of section 0.1, let

$$w(x,y) = \int_y^H [\tilde{h}(x,t) - t]\,dt ,$$

where

$$\tilde{h}(x, y) = \begin{cases} h(x, y), & \text{if } (x, y) \in \mathcal{D}, \\ y, & \text{otherwise}. \end{cases}$$

Let Ω be the rectangle ABCF (see Figure 1 of section 0.1) and let g be defined on the boundary $\partial\Omega$ of Ω as follows:

$$\begin{aligned} g &= (H - y)^2/2, \quad \text{on AB}, \\ &= H^2/2 + (h_d^2 - H^2)x/2\ell, \quad \text{on BC}, \\ &= (h_d - y)^2/2, \quad \text{on CD}, \\ &= 0, \quad \text{on DEFA}. \end{aligned}$$

It can be shown that w solves the problem: Minimize the integral

$$J(v) = \frac{1}{2} \iint_\Omega (v_x^2 + v_y^2 + 2v)\,dxdy,$$

among the class of functions v which are defined and non-negative on Ω and equal to g on $\partial\Omega$. This problem is one of several equivalent formulations of variational inequalities.

The use of variational inequalities has led to profound results. Further details will be found in the papers of Baiocchi listed in the bibliography. Variational inequalities can be solved numerically by several methods including finite differences and finite elements.

<u>Trial free boundary methods</u>

We repeat the basic FBP:

$$\mathcal{A}u = 0, \quad \text{in } \mathcal{D},$$

$$\mathcal{B}u = 0, \quad \text{on } \partial\mathcal{D},$$

$$\mathcal{C}u = 0, \quad \text{on } \Gamma.$$

One approach to solving this FBP numerically is to generate a sequence of trial FBS $\Gamma^{(k)}$ and approximate solutions $u_h^{(k)}$ as follows:

Step 0. Guess an initial trial FB, $\Gamma^{(o)}$ say.

Step 1. Given $\Gamma^{(k)}$ let $\mathscr{D}^{(k)}$ be the corresponding domain. Compute an approximation, $u_h^{(k)}$ say, to the solution $u^{(k)}$ of the problem

$$\mathscr{A}u^{(k)} = 0, \quad \text{in} \quad \mathscr{D}^{(k)},$$

$$\mathscr{B}u^{(k)} = 0, \quad \text{on} \quad \partial\mathscr{D}^{(k)}.$$

Step 2. Given $\Gamma^{(k)}$ and $u_h^{(k)}$ compute a new trial FB $\Gamma^{(k+1)}$ by requiring that $\mathscr{C}u_h^{(k)}$ should be approximately equal to zero on $\Gamma^{(k+1)}$; i.e. "move the boundary" from $\Gamma^{(k)}$ to $\Gamma^{(k+1)}$.

Following Birkhoff [1961] we will call this method, and variations thereof, a trial free boundary method. (Previously, we have called methods of this type "move-the-boundary" methods (Cryer [1970a]), but Birkhoff's terminology is better.)

In Step 2 the computation of the approximate solution $u^{(k)}$ requires the numerical solution of a fixed boundary value problem, which can be done using any standard method. We then speak of a trial free boundary method using finite elements, a trial free boundary method using finite differences, etc.

Cryer [1970a] gives a general discussion of trial free boundary methods.

Time-dependent methods

Porous flow FBPS have often been solved by numerically solving a time-dependent porous flow problem and letting the time become very large, so that steady-state is approached. We call such methods time-dependent methods, and qualify them by indicating the numerical method used to solve the time-dependent problem.

1. The Governing Equations

Most of the FBPS considered in the literature concern the flow subject to Darcy's law of an incompressible heavy fluid through an isotropic homogeneous medium, and the reader should assume that this case is being considered unless it is explicitly stated otherwise. The governing equations for this special case are given in section 1.1.

There are minor but extremely irritating differences of notation in the literature. The following notation will be used here:

$\underline{x}$	co-ordinate vector
$x = x_1$ and $y = x_2$	plane co-ordinates
r and y or x and y	cylindrical co-ordinates
$\underline{q}$	fluid velocity
u and v	velocity components
ρ	density of fluid
p	fluid pressure
$p_a = 0$	atmospheric pressure
g	gravity acceleration (along negative y-axis)
$\underline{g}$	gravity vector
ϕ	velocity potential
ψ	stream function (in saturated flow)
$\psi = p/\gamma = p/\rho g$	suction (in unsaturated flow)
$k = \mu K/\rho g$	permeability

$\underline{k} = (k_{ij}) = \mu\underline{K}/\rho g$ permeability tensor

$K = k\rho g/\mu$ hydraulic conductivity

$\underline{K} = (K_{ij}) = \underline{k}\rho g/\mu$ hydraulic conductivity tensor

μ dynamic viscosity

$h = y + \frac{1}{g}\int \frac{dp}{\rho(p)}$ hydraulic head (or piezometric head); this is often denoted by φ or ϕ

θ moisture content

$\gamma = \rho g$ specific weight of water

It is difficult to formulate the governing equations for porous flow for several reasons:

(i) The flow of a fluid through a porous medium, for example the flow of water through soil, is an extremely complex matter. The ground consists of layers of soil resting on layers of rock, and is certainly not homogeneous. Furthermore, the different layers are often not isotropic. (To mention but one of the many possible causes of anisotropy, if a layer of soil has been formed by sedimentation then flat particles tend to be oriented with their longest dimensions parallel to the layer.) At the microscopic level soil consists of a large number of soil particles of different sizes and shapes, the interstices between the particles being filled with water and air. As water flows through soil it is subject to electrical and chemical processes which are not fully understood. For a detailed discussion of soil physics see Childs [1969], Bear [1972], Kirkham and Powers [1972].

(ii) It is difficult to perform accurate experiments for porous flow problems. It is, for example, a non-trivial matter to obtain representative and reproducible soil samples. Several years ago we were directly involved in some porous flow experiments and were surprised at the bulk of the probe which was used to measure the pore water pressure p. In the FBPS discussed in this chapter the FB is shown as a smooth curve. In contrast, experimentally observed FBS are often rather irregular. We have seen this in the laboratory, and it can, for example, also be seen in the figures in the fundamental work of L. Casagrande [1932, 1934] on the flow through dams.

(iii) The theory of porous flow is not supported by an underlying theory in the way that, say, the theory of heat conduction is supported by the theory of thermodynamics.

(iv) There have been many attempts to derive the global (macroscopic) governing equations by making assumptions about the local (microscopic) equations and the structure of the porous medium (Bear [1972, p. 161], Scheidegger [1963, p. 641]). In the simplest model it is assumed that the porous medium consists of a large number of circular capillary tubes and that the fluid is a Newtonian fluid. Much more complicated models have been considered; Scheidegger [1966] gives a very readable concise review and Bear [1972, p. 161] gives more details and describes recent work. It seems fair to say, however, that while these models provide insight none of them is completely satisfactory.

As a result of the difficulties mentioned above there is agreement about the governing equations for linear isotropic homogeneous saturated flow, but there is not universal agreement about the correct generalizations of these equations to more complicated flows.

The great majority of the porous flow FBPS which have been considered in the literature are for linear isotropic homogeneous saturated flow. Recently, however, some nonlinear, anisotropic, or inhomogeneous FBPS have been solved (numerically); we believe that such FBPS will be increasingly studied and therefore discuss their governing equations.

1.1. <u>Linear incompressible saturated flow in an isotropic homogeneous medium</u>

The first basic equation is the equation of continuity

$$\operatorname{div} \underset{\sim}{q} = 0 .$$

The equation of continuity permits the introduction of a stream function ψ in plane and axially symmetric problems:

$$\text{(P)} \qquad u = \psi_y, \qquad v = -\psi_x,$$

$$\text{(A)} \qquad u = \frac{1}{x}\psi_y, \quad v = -\frac{1}{x}\psi_x .$$

The second basic equation expresses Darcy's law that the flow velocity is proportional to the gradient of the hydraulic head:

$$\underline{q} = -K \operatorname{grad} h$$

where the scalar constant K is the <u>hydraulic conductivity</u> of the medium.

Equivalently,

$$\underline{q} = -\frac{K}{\rho g} \operatorname{grad}(p + \rho g y) = -\frac{k}{\mu} \operatorname{grad}(p+\rho g y) ,$$

where the scalar constant $k = \mu K/\rho g$ is the <u>permeability.</u>

It follows from Darcy's law that the function

$$\phi = -Kh = -\frac{k}{\mu}(p+\rho g y)$$

is a velocity potential:

$$\underline{q} = \operatorname{grad} \phi ,$$

$$\text{(P)} \quad u = \phi_x, \qquad v = \phi_y,$$

$$\text{(A)} \quad u = \phi_x, \qquad v = \phi_y .$$

In the literature, the negative velocity potential (or specific discharge potential) is often used.

It follows immediately that in plane and axially symmetric problems:

$$\left.\begin{aligned} &\text{(P)} \quad \phi_{xx} + \phi_{yy} = 0, \\ &\text{(A)} \quad \phi_{xx} + \frac{1}{x}\phi_x + \psi_{yy} = 0, \end{aligned}\right\}$$

$$\left.\begin{aligned} &\text{(P)} \quad \psi_{xx} + \psi_{yy} = 0, \\ &\text{(A)} \quad \psi_{xx} - \frac{1}{x}\psi_x + \psi_{yy} = 0. \end{aligned}\right\}$$

These equations form the basis of most work on porous flow FBPS.

1.2. Linear compressible anisotropic saturated flow

There are three basic equations: the equation of continuity; the equation of state; and Darcy's law.

The equation of continuity is

$$\operatorname{div}(\rho \underset{\sim}{q}) = 0.$$

As in fluid mechanics the equation of continuity permits the introduction of a stream function ψ in plane and axially symmetric problems:

$$\left.\begin{aligned} &(P)\quad \rho u = \psi_y, \quad \rho v = -\psi_x, \\ &(A)\quad \rho u = \frac{1}{x}\psi_y, \quad \rho v = -\frac{1}{x}\psi_x. \end{aligned}\right\}$$

The equation of state expresses the density ρ as a function of the pressure p. To a very good approximation, water is incompressible. However, when studying the flow of gases through soil, as is necessary when considering natural gas wells, compressibility must be taken into account. The equation of state is usually taken to be of the form (Muskat [1937])

$$\rho = \rho(p) = \rho_0 p^m e^{\beta p},$$

where ρ_0, m, and β are constants. Particular cases are:

Liquids, incompressible: $m = 0, \beta = 0$,

Liquids, compressible: $m = 0, \beta > 0$,

Gases, isothermal: $m = 1, \beta = 0$,

Gases, adiabatic: $m > 1, \beta = 0$.

The linear form of Darcy's law which is most often used is (Scheidegger [1963, p. 636], Bear [1972, p. 124, 136]),

$$\underline{q} = -\underline{K}(\underline{x}) \text{ grad } h$$

where

$$h = y + \frac{1}{g} \int \frac{dp}{\rho(p)}$$

and $\underline{K}(\underline{x})$ is a symmetric tensor, the <u>hydraulic conductivity tensor</u>,

$$\underline{K}(\underline{x}) = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} .$$

In plane problems $\underline{K}$ must be of the form

$$\underline{K}(\underline{x}) = \begin{bmatrix} K_{11}(\underline{x}) & K_{12}(\underline{x}) \\ K_{21}(\underline{x}) & K_{22}(\underline{x}) \end{bmatrix},$$

with $K_{12} = K_{21}$.

It should be observed that there is apparently no fundamental physical law which would require $\underline{K}(\underline{x})$ to be symmetric. Flügge [1972, p. 101] observes that he tried to find an example of a porous flow system with unsymmetric $\underline{K}(\underline{x})$ but did not succeed. It may be remarked that in the case of heat conduction the thermal conductivity $\underline{\kappa}$ plays a similar role to that of the hydraulic conductivity $\underline{K}$ in porous flow. It is a consequence of the fact that entropy cannot decrease, that the thermal conductivity $\underline{\kappa}$ is symmetric and positive semi-definite (Truesdell and Toupin [1960, p. 709], Carslaw and Jaeger [1959]).

It is of interest that inhomogeneity is sometimes desirable. Kealy and Soderberg [1969, p. 30] make the following recommendation for mill-tailing dams: "Never construct homogeneous dams. Zoned-type construction must be used to control seepage water and consequently increase stability".

1.3. Nonlinear saturated flow

The governing equations discussed above assumed that there was a linear relationship (Darcy's law) between the gradient of the piezometric head, grad h, and the velocity $\underline{q}$. However, it has been found experimentally that the relationship between grad h and $\underline{q}$ is nonlinear when the flow rate is very low or very high. The various nonlinear generalizations of Darcy's law are described below.

1.3.1. Low speed flow

It has been found that there is very little flow when $J = |\text{grad } h|$ is less than a constant J_0, the threshold gradient. This may be thought of as being due to intermolecular attractive forces which are negligible at higher flow rates.

Various modifications of Darcy's law have been suggested. Many of the modifications are of the form

$$\underline{q} = \begin{cases} 0, & \text{if } J \leq J_0 \\ -KF(J_0/J) \text{ grad } h, & \text{if } J > J_0, \end{cases}$$

where $F(t)$ is a given function. Among the suggestions for $F(t)$ we mention:

$F(t) = 1-t$, (Irmay; see Bear [1972, p. 127]),

$$F(t) = (1-t)^2 - \frac{2}{3} t[(1-t)^{3/2} \text{ arc tanh } (1-t)^{1/2} - \ln t - (1-t)],$$

(Kovacs [1969]) .

For further references see Bear [1972, p. 128].

The concept of a threshold gradient introduces a new FB namely the FB separating the region of flow where $J \leq J_0$ from the region of flow where $J > J_0$. So far as we are aware, this FBP has not been considered in the literature.

1.3.2. High speed flow

At high flow rates it is found that the flow rate is less than that predicted by Darcy's law. This can be attributed to the onset of turbulence which causes increased energy losses.

The most general generalization of Darcy's law is that of Engelund:

$$\text{grad } h = -F(|\underline{q}|)\underline{q}$$

where $F(t)$ is a given function. Scheidegger [1960], Ahmed and Sunada [1969], and Bear [1972, p. 182] summarize the various choices for $F(t)$ which have been suggested. Almost all the proposed functions are of the form

$$F(t) = a + bt^{m-1} \qquad \text{(Forchheimer)}$$

where a, b, and m are constants.

An important application of the nonlinear versions of Darcy's law arises in connection with porous flow through rockfill dams, that is dams built with rocks rather than soil. The flow through rockfill dams is apparently more turbulent than the flow through corresponding earth dams, and it is necessary to use a nonlinear law. Fenton [1972a] has used

$$F(t) = bt^{m-1} ,$$

with $m = 1, 2,$ and 1.86. Volker [1969] has used

$$F(t) = .319 + 11.821\, t$$

and

$$F(t) = 8.893\, t^{.745} ;$$

Volker obtained the values of the parameters by fitting to experimental data using least-squares, and he observes that both forms of $F(t)$ fitted the data accurately.

An interesting result of Volker's work is that the form of the FB does not depend very much upon the choice of $F(t)$, but that the nonlinear equations give much more accurate results for the discharge through the dam, (Volker [1969, p. 2108 and p. 2110]).

Hamel [1934, p. 156] discusses modifications to handle infinite speeds at points of discontinuity.

1.3.3. Slip flow

In low pressure gas flow it is found that the flow rates are higher than theoretically predicted. This is called the slip phenomenon or Klinkenberg effect.

Klinkenberg has suggested that (Bear [1972, p. 128])

$$k_g = k_\ell(1 + b(k_\ell)/p)$$

where k_g and k_ℓ are the permeabilities to gas and liquids, respectively. If the pressure p varies substantially then this implies that k_g varies, but we do not know whether this has been taken into account in the literature.

1.4. Partially saturated flow

In the discussion above of the flow of water through soil two assumptions were made: (i) the flow is saturated, that is, the soil is either wet or dry; (ii) the motion of the air in the dry soil is negligible. In certain circumstances these assumptions are not reasonable, and it is necessary to consider the movement of water and air through the soil.

In the general case of multi-phase flow the porous medium contains m fluids which flow through the medium; this is discussed in section 1.5. An important special case of two-phase flow is unsaturated flow which is often considered in the water/air flow problems of irrigation. In unsaturated flow problems the motion of the air is neglected, but allowance is made for the presence of air.

Unsaturated flow involves the introduction of a new variable θ, the volume of water per unit volume of soil. For steady-state unsaturated flow of water the governing equations are (Braester, Dagan, Neuman and Zaslavsky [1971, p. 14]):

$$\operatorname{div} \underline{q} = 0,$$

$$\underline{q} = -K \operatorname{grad} h,$$

$$h = y + p/\rho g = y + p/\gamma ,$$

where, in distinction to the case of saturated flow, K is a function of the present and past states of the flow.

It has been found experimentally that K is a highly nonlinear function which exhibits hysteresis and the relationships between K, θ, and $\psi = p/\gamma$ have been stated in several ways. (E. E. Miller and Klute [1967], Braester, Dagan, Neuman, and Zaslavsky [1971]). In the applications of interest to us, K and θ have been expressed as functions of the pressure p. For $p \geq p_a = 0$ K and θ have been assumed to be constant. For $p \leq 0$ expressions of the form

$$K = \frac{K_0}{A(-p)^{\alpha} + 1},$$

$$\theta = \frac{\theta_0}{B(-p)^{\beta} + 1}, \tag{1}$$

have been used, where K_0, θ_0, A, and B, are constants.

In partially saturated flows there is no water/air FB: as can be seen from equation (1), the water content is never zero. However, the length of seepage surfaces is unknown because seepage occurs only when the pressure p is positive (see section 2.1).

A simplification in partially saturated flow which is sometimes made is that *the region of unsaturated flow is confined to a narrow zone above the water table* $(p = 0)$ called the capillary fringe (Bear [1972; pp. 259, 262, and 480]). The term "capillary fringe" is used because the fringe is thought of as being produced by capillary forces (in analogy with the rise of a liquid in a capillary tube). In saturated/capillary flow the following assumptions are made:

(i) There are three regions - the saturated region, the capillary region, and the dry region.

(ii) The governing equations are the same in the capillary region and the saturated region. The dry/capillary interface is a FB. The additional condition on Γ is that

$$p = -p_c$$

where the constant p_c is the capillary pressure.

Bear [1972, p. 259] states that when a capillary fringe is used, the completely saturated region extends to the capillary/dry interface. It seems to us that this approach can lead to confusion because although the governing equations are the same in the saturated region and the capillary region the boundary conditions are different (see section 2.1).

When comparing saturated flows with partially saturated flows or saturated/capillary flows, it is reasonable to compare the water/air FB in the saturated flow with the water table, that is, the curve $p = 0$, in the other flows.

It is reasonable to assume (although not proved) that as the constants A and B in equation (1) tend to ∞ the solutions of the partially saturated flow problem will converge to the solution of the corresponding saturated flow problem. Thus, partially unsaturated flows provide a means for approximating saturated flow FBPS by nonlinear

fixed boundary value problems. This approach is related to the idea of approximating variational inequalities by nonlinear fixed boundary value problems.

For further information about unsaturated flow the reader should consult Braester, Dagan, Neuman, and Zaslavsky [1971]. Freeze [1971] gives a general discussion of the numerical solution of the three-dimensional partially saturated problems. In addition we draw attention to some work on the numerical solution of time-dependent unsaturated flows which is not quoted by Braester et al: Amerman [1969], Carroll [1969].

See also: Rubin [1968], Verma and Brutsaert [1970], Jury [1973].

1.5. Multi-phase porous flow

In multi-phase porous flow the porous medium contains $m \geq 2$ fluids. For example, in oil reservoirs the simultaneous flow of oil, gas, and water is encountered. In irrigation problems, the water available to plants is in the form of water vapour, so that the two-phase water/water-vapour problem is of interest (see E. E. Miller and Klute [1967]).

For a general discussion see Bear [1972, chapter 9]. For numerical solutions see Startzman [1969].

1.6. Electrokinetic porous flow

The subject of electrokinetic porous flow is concerned with the flow of a fluid in a porous medium under the influence of the hydrodynamic and electric fields.

In plane problems the governing equations are as follows:

$$v_x = k_{hx} \frac{\partial h}{\partial x} + k_{ex} \frac{\partial E}{\partial x} ,$$
$$v_y = k_{hy} \frac{\partial h}{\partial y} + k_{ey} \frac{\partial E}{\partial y} , \tag{1}$$

$$j_x = G_x \frac{\partial h}{\partial x} + \frac{1}{\rho x} \frac{\partial E}{\partial x} ,$$
$$j_y = G_y \frac{\partial h}{\partial y} + \frac{1}{\rho y} \frac{\partial E}{\partial y} , \tag{2}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 ,$$
$$\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} = 0 , \tag{3}$$

Here, v_x and v_y are the velocity components of the fluid, j_x and j_y are the current densities, h is the hydraulic head, E is the electric intensity, and k_{hx}, k_{hy}, k_{ex}, k_{ey}, G_x, G_y, ρ_x, ρ_y, are constants.

Equations (1) are a modification of Darcy's law, equations (2) are a modification of Ohm's law, and equations (3) express the conservation of mass and current.

The above equations are given by Lewis and Garner [1972] who give further references. Lewis and Garner solve a fixed boundary electrokinetic porous flow problem using finite elements. Lewis and Humpheson [1973] solve a time-dependent electrokinetic porous flow well problem; see section 3.4.1.1.

1.7. Plant roots

The flow of water to plant roots is of importance in agriculture; indeed, it is the main purpose of irrigation to provide an adequate flow of water.

Two approaches have been used to apply the theory of porous flow to water uptake by roots (E. E. Miller and Klute [1967, p. 239]):

(i) Root system model

The plant roots are assumed to be distributed continuously throughout the soil and the uptake of water is treated by replacing the equation of continuity by a modified equation

$$\operatorname{div} \underset{\sim}{q} + G = 0,$$

where G is negative and represents the uptake of water by the roots per unit volume of soil.

(ii) Single-root model

The details of the flow about a single root are examined. It is customary to assume that the root is cylindrical, and that the water flows through the root wall at a given rate. The boundary condition is thus

$$\underline{q} \equiv -K \text{ grad } h = \text{constant},$$

on the root wall $x = a$.

It appears that very little work has been done on this interesting problem. The two approaches described above are rather rudimentary. A more sophisticated approach would require an understanding of the mechanism of water absorption by roots, and Slatyer [1960] is a good starting point for a study of the literature on this subject (see also Hagan, Haise, and Edminster [1967, section VI]).

2. Boundary conditions

2.1. Boundary conditions: Fixed boundaries

The most common fixed boundary conditions are given immediately below and are illustrated in Figure 1 for the case of water seeping through an earth dam. Unless otherwise stated, saturated flow is considered.

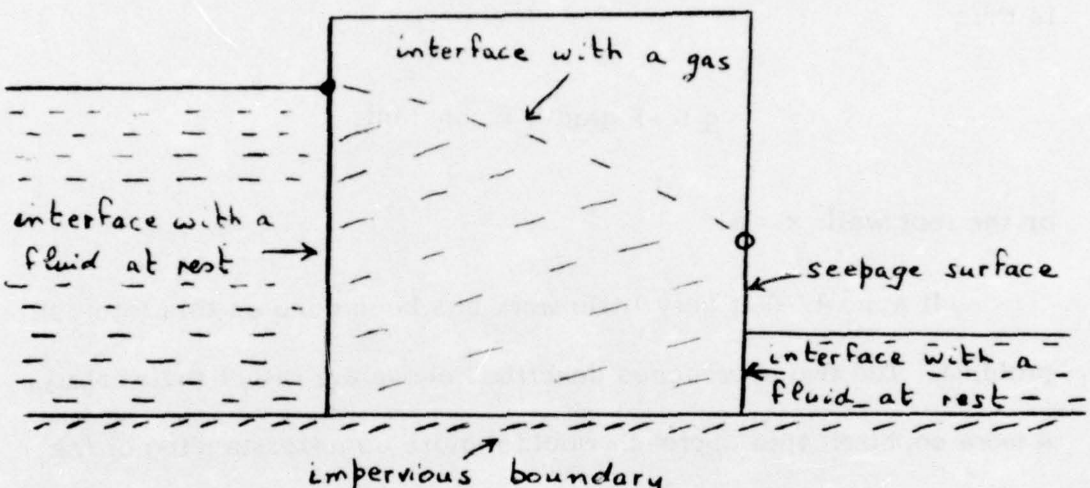


Figure 1: Common boundary conditions in porous flow

(i) Impervious boundary

An impervious boundary arises at the surface of the rock layer or retaining wall. No flow occurs across the boundary and the boundary condition is:

$$\underline{q} \cdot \underline{n} = 0.$$

If Darcy's law holds then

$$\underline{k} \text{ grad } h \cdot \underline{n} = 0.$$

In problems for which a velocity potential ϕ exists, we have

$$\phi_n = \frac{\partial \phi}{\partial n} = 0 ,$$

and in problems for which a stream function ψ exists, we have

$$\psi = \text{constant} .$$

These boundary conditions hold for saturated, partly saturated, and saturated/capillary flow.

(ii) Interface with fluid at rest

If the fluid at rest is denoted by a subscript 2, then since it is at rest the hydrostatic head is constant,

$$h_2 = y + \frac{1}{g} \int_0^p \frac{dp}{\rho_2(p)} = \text{constant} . \qquad (1)$$

This is an implicit equation which can be solved for the pressure p_2,

$$p_2 = p_2(y) .$$

At the interface the pressures must be equal so that the boundary condition is

$$p = p_1 = p_2(y),$$

or, equivalently,

$$h = h_1 = y + \frac{1}{g}\int_0^{p_2(y)} \frac{dp}{\rho_1(p)},$$

where a subscript 1, or no subscript, denotes the fluid in motion.

So far as we are aware, boundary conditions of this generality have not been considered in the literature.

The constant h_2 is usually determined from a given value of $p_2(y)$. In particular, if the fluid 2 has an interface with air at a height $y = H$ then

$$h_2 = H + \frac{1}{g}\int_0^{p_a} \frac{dp}{\rho_2(p)} = H.$$

Three special cases have been considered:

(ii)(a) <u>Interface with gas</u>

At a gas interface $\rho_2(p)$ is so small that the condition h_2 = constant may be approximated to high accuracy by the condition p_2 = constant.

The most common gas interface is with air for which we have

$$p_2 = p_a = 0,$$

so that the boundary condition becomes

$$p = 0,$$

or, equivalently,

$$h = y.$$

(ii)(b) Interface between incompressible liquids

If both fluids are incompressible then

$$h_2 = y + p_2/\rho_2 g,$$

so that

$$p = \rho_2 g(h_2 - y),$$

or, equivalently,

$$h = y + \frac{\rho_2}{\rho_1}(h_2 - y).$$

(ii)(c) Water interface with reservoirs, wells, head waters, or tail waters

This is a special case of (b) above with $\rho_1 = \rho_2 = \rho$ so that

$$p = \rho g(h_2 - y),$$

or, equivalently,

$$h = h_2 .$$

(iii) Seepage surface

When a FB approaches a fixed impermeable boundary or a downstream tailwater it cannot intersect them. Instead, a seepage surface forms (see Figure 1). On a seepage surface the fluid is in contact with the air so that, as in (ii)(a) above, we have

$$p = 0 ,$$

or, equivalently,

$$h = y .$$

In partially saturated flow a seepage surface occurs when the pressure p is greater than atmospheric pressure, so that the length of the seepage surface is unknown a-priori (see Freeze [1971a]).

(iv) Exposed capillary surface or unsaturated surface

In a region of unsaturated flow, or in a capillary fringe, the pressure p is less than atmospheric and it is usually assumed that the fluid cannot flow across an exposed surface. Such exposed surfaces are thus usually treated as being equivalent to impervious boundaries and the boundary condition is that

$$\underline{q} \cdot \underline{n} = 0.$$

For equivalent forms see (i) above.

(v) Prescribed heads

In certain problems, notably problems involving flow towards wells, it is assumed that the head h is a given function on part of the boundary; usually h is assumed to be constant.

(vi) Prescribed flow rate

Sometimes the flow rate $\underline{q} \cdot \underline{n}$ is prescribed on part of the boundary. An interesting example arises in the study of the absorption of water by plant roots; Gardner has analysed this problem on the assumption that the root absorbs moisture at a constant rate (see E. E. Miller and Klute [1967, p. 238]).

(vii) Drains

The pressure p is usually prescribed at a drain . In most problems $p = p_a = 0$.

(viii) Semi-pervious boundaries

If two porous media are separated by a thin layer of semi-porous material, then it is often assumed that

$$\underline{q} \cdot \underline{n} = (h_1 - h_2)$$

where h_1 and h_2 denote the heads in the two media. See Bear [1972, p. 216] and Hantush [1964].

The asymptotic behavior of the solution at the intersection of different boundaries is considered by Kravtchenko, de Saint-Marc, and Boreli [1955], Polubarinova-Kochina [1962, p. 276].

In conclusion we observe that infinite flow velocities occur in the following circumstances (Muskat [1937, p. 291], Bear [1972, p. 289]):

(a) At the point of intersection of a seepage surface with a water-table.

(b) When the interface with an upstream or downstream reservoir has a corner with interior angle (in the flow domain) greater than π.

(c) When impervious boundaries, seepage surfaces, or capillary exposed surfaces have corners.

So far as we are aware, only Hamel [1934, p. 156] has explicitly considered the errors caused by assuming the validity of Darcy's law at all flow velocities. The various high speed flow formulas discussed in section 1.3.2 automatically prevent infinite velocities.

2.2. Boundary conditions: FBS

It is possible to conceive of very complicated porous flow FBPS. For example, one could consider the breaking of an earth dam due to excess pore water pressure, in which case the dam faces would be FBS. However, so far as we are aware, the porous flow FBPS which have been treated in the literature involve only fluid interface FBS between two fluids denoted by the superscript i for $i = 1, 2$. Two basic boundary conditions are imposed, namely continuity of flow and continuity of pressure:

(i) Continuity of Flow

The following cases arise:

(a) No flow. The usual condition at a fluid interface is

that there should be no flow across the interface. The boundary conditions are thus the same as for an impervious boundary (see section 2.1) namely:

$$\underline{q}^{(i)} \cdot \underline{n} = 0, \quad \text{for } i = 1, 2.$$

If Darcy's law holds then

$$\underline{K}^{(i)} \operatorname{grad} h^{(i)} \cdot \underline{n} = 0, \quad \text{for } i = 1, 2.$$

If velocity potentials $\phi^{(i)}$ exist then

$$\phi_n^{(i)} = 0, \quad \text{for } i = 1, 2$$

and if stream functions $\psi^{(i)}$ exist then

$$\psi^{(i)} = c^{(i)} = \text{constant}, \quad \text{for } i = 1, 2.$$

(b) Accretion. It is sometimes assumed that there is accretion on the FB, and this is expressed by the condition

$$\underline{q}^{(1)} \cdot \underline{n} = \underline{N} \cdot \underline{n}, \tag{1}$$

where $\underline{N}$ is a specified flow. Usually $\underline{N}$ is of the form

$$\underline{N} = - N\underline{e}_y \, .$$

where $\underline{e}_y$ denotes the unit vector in the positive y direction, $N > 0$ corresponds to positive accretion

(such as rainfall or irrigation) and $N < 0$ corresponds to negative accretion (such as evaporation). (Bear [1972, p. 256]).

The form of (1) is somewhat arbitrary. One might perhaps assume that in evaporation the rate of evaporation is proportional to the exposed area so that (1) is replaced by

$$\underline{q} \cdot \underline{n} = -N > 0 \; .$$

However, the boundary condition (1) is generally accepted.

Pozzi [1974a, p. 24] considers evaporation from a dam and observes that the problem may not be well-posed if the rate of evaporation (using (1)) is too large.

(ii) Continuity of pressure

The second condition at a fluid interface is that the pressure should be continuous,

$$p^{(1)} = p^{(2)} \; .$$

The following cases arise:

(a) Interface with a fluid at rest

This is discussed in detail in section 2.1. The most common cases are:

(α) Water/air interface for which

$$p^{(1)} \equiv p_{water} = p_a = 0;$$

or equivalently,

$$h^{(1)} = y.$$

(β) Fresh water/salt water interface for which

$$p^{(1)} \equiv p_{fw}$$

$$= \rho_{sw} g(h_{sw} - y),$$

or, equivalently,

$$h^{(1)} \equiv h_{fw} = y + \frac{\rho_{sw}}{\rho_{fw}} (h_{sw} - y),$$

where the salt water hydraulic head h_{sw} is a constant.

(γ) Capillary fringe

A capillary fringe is an approximation to partially saturated flow (see section 1.4). It is assumed that the air/water interface occurs not when the pressure is zero but when the pressure is equal to $-p_c$ where $p_c > 0$ is a constant, the capillary pressure.

It is sometimes possible to combine the above equations in a useful way. Consider the conditions at an air/water interface with accretion, namely

$$h = y, \tag{3}$$

$$\underline{q} \cdot \underline{n} = -N\underline{e}_y \cdot \underline{n}\,, \tag{4}$$

Bear [1972, p. 257] shows that if Darcy's law is of the form

$$\underline{q} = -\begin{pmatrix} K_{11} & 0 \\ 0 & K_{22} \end{pmatrix} \text{grad } h$$

then (3) and (4) lead to the equation

$$K_{11}\left(\frac{\partial h}{\partial x}\right)^2 + K_{22}\left(\frac{\partial h}{\partial y} - \frac{K_{22} + N}{2K_{22}}\right)^2 = \left(\frac{K_{22} - N}{2K_{22}}\right)^2 K_{22}\,. \tag{5}$$

Mauersberger [1965a] shows that if $N = 0$ and Darcy's law is of the form

$$\underline{q} = -\underline{K} \text{ grad } h$$

then (3) and (4) lead to

$$\text{grad } h \cdot \underline{K} \text{ grad } h - \underline{e}_y \cdot \underline{K} \text{ grad } h = 0, \tag{6}$$

and also Mauersberger asserts that (3) and (4) are equivalent to (3) and (6). We believe that this assertion of Mauersberger can be rigorously justified with the use of the maximum principle.

2.2.1. Points of detachment

The behavior of a FB near a point of detachment is discussed by Muskat [1937, p. 287], A. Casagrande [1940, p. 302], Harr [1962, p. 21] and Bear [1972, p. 284].

The analysis of the point of detachment was first systematically investigated by A. Casagrande [1940]. There are a number of subtle difficulties, and in the literature the results of Casagrande are often quoted without proof, reference being made to Casagrande's paper. This is not entirely satisfactory since Casagrande used an engineering approach. The best treatment in the literature is that of Bear [1972, p. 284] who makes systematic use of the hodograph plane.

In the Figures below we show graphically various possible configurations for linear homogeneous isotropic saturated flow; we believe that these are correct but, (as explained at the end of this section) we are not entirely satisfied with all the justifications which have been advanced in the literature. For clarity, the curvature of the FB is neglected in the Figures. The flow velocity and slope of the FB at the point of separation are denoted by q and β, respectively. The slope β can usually be found from the condition that the flow velocity must be finite at the point of detachment.

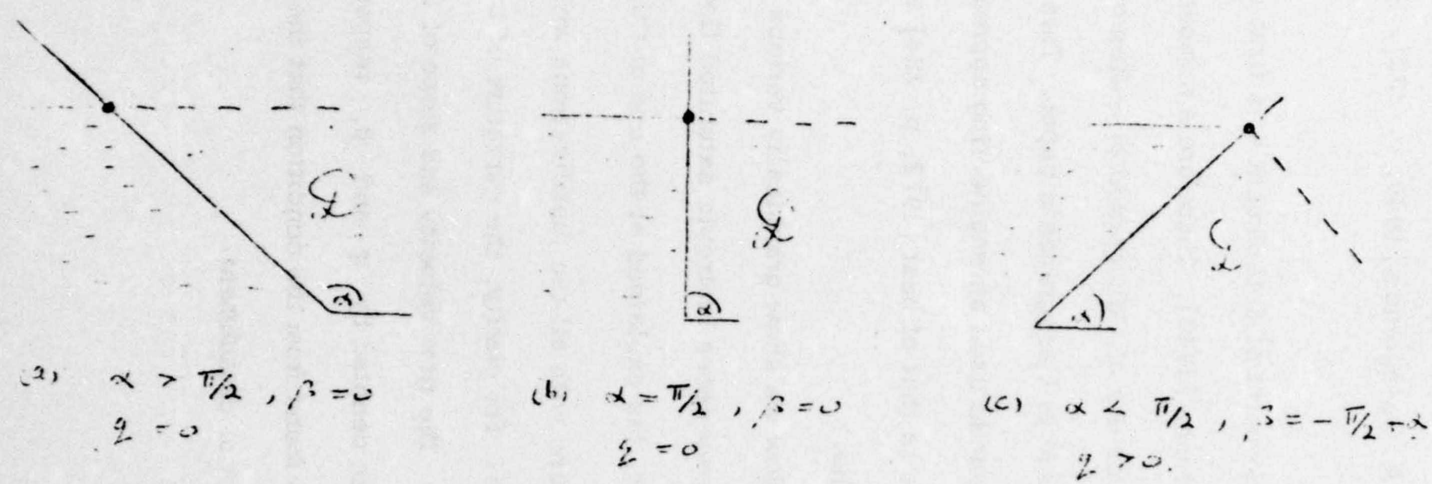


Figure 1: Intersection of a FB with an upstream water table (no accretion)

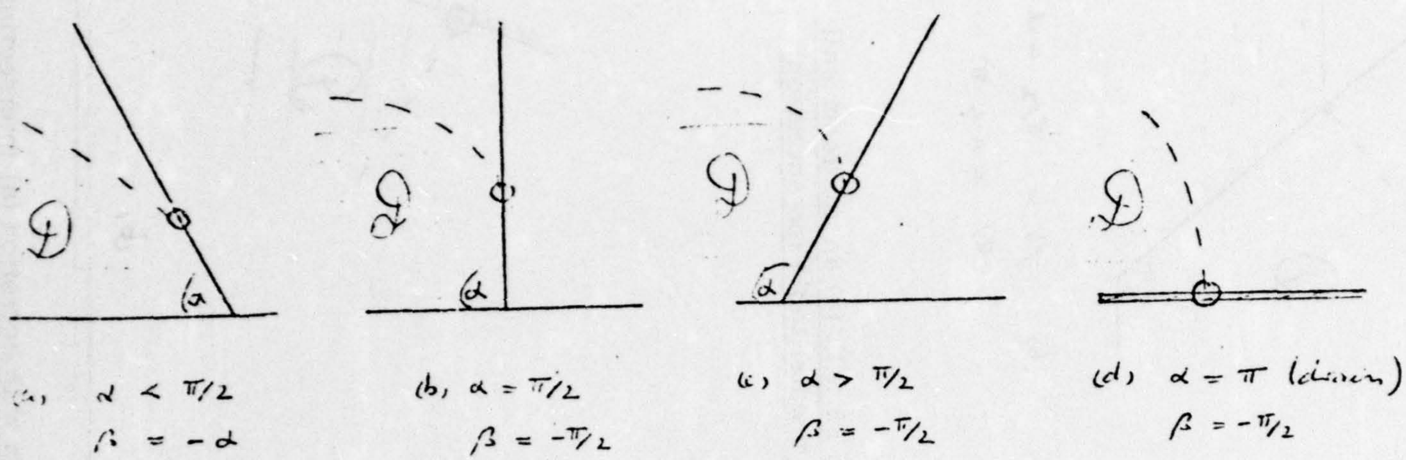
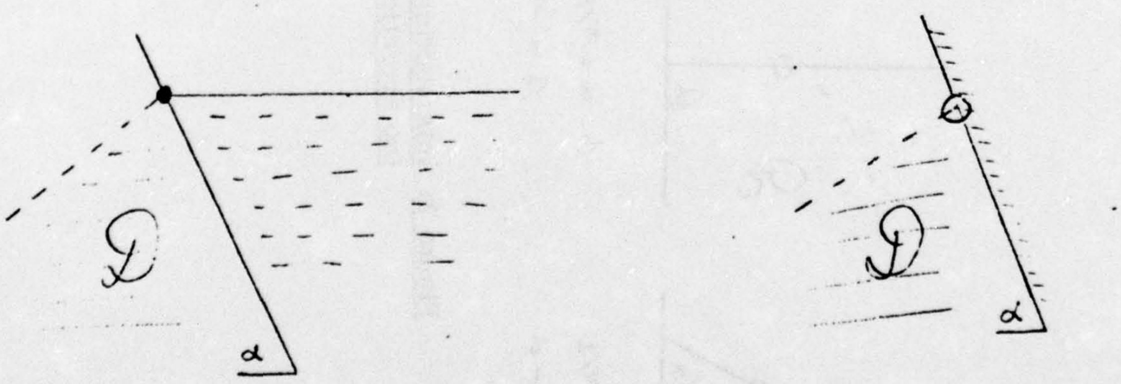


Figure 2: Intersection of a FB with a downstream seepage surface (no accretion)

(a) $0 < \alpha < \pi$
$\beta = 0$
$q = 0$

(b) $\beta = \pi/2 - \alpha$
$\pi/2 < \alpha < \pi$

Figure 3: Intersection of a FB with (a) an upstream wall
(b) a downstream water table (no accretion)



(a) $0 < \alpha \leq \pi/2$

(b)

Figure 4: Possible effects with accretion (a) intersection with downstream water-table, (b) intersection with downstream wall

The above Figures do not exhaust all the possibilities. If accretion is present (negative or positive) then slight modifications of Figures 1, 2, and 3 are necessary; see Bear [1972, p. 285 and p. 289].

The asymptotic behaviour of the FB near a point of detachment is discussed by Kravtchenko, de Saint-Marc, and Boreli [1955], J. M. Taylor [1971] and Aitchison [1972].

We now consider some of the shortcomings of the arguments associated with the geometries shown in Figures 1 to 4:

(i) One argument used to justify the existence of seepage surfaces is that if a seepage surface did not exist then the FB would intersect the water-table and infinite velocities would occur (Muskat [1937, p. 289], Bear [1972, p. 260]). However, the introduction of a seepage surface does not prevent infinite velocities, because the flow velocity is infinite at the intersection of a seepage surface with a water table (Bear [1972, p. 287], Muskat [1937, p. 291]).

(ii) It is assumed that, in the absence of accretion, a FB slopes downwards in the direction of flow. That is, if the flow is in the positive x-direction and the FB is the curve $y = y(x)$ then $dy/dx \leq 0$. This is of course a reasonable assumption but the proof (using the maximum principle) is seldom given (see Shaw and Southwell [1941, p. 15]). This assumption is used to exclude certain geometries:

(a) To explain why there is a downstream seepage surface but not an upstream seepage surface (Bear [1972, p. 260]).

(b) To exclude the possibility that at an overhanging interface with an upstream water-table (Figure 1a) the FB is normal to the interface, that is, $\beta = \alpha - \pi/2$.

(iii) The arguments for the existence of a vertical seepage surface at an overhanging interface with a downstream water-table (Figure 2c) are rather involved. The arguments of Bear [1972, p. 260] do not hold in this case. Muskat [1937, p. 290] states that he cannot justify the existence of the seepage surface mathematically except in the case when there is no downstream water.

In conclusion we observe that we believe that the behavior shown in the above Figures is correct. However a clear statement of the mathematical assumptions involved is desirable. Our criticisms are not just a demand for "proofs of obvious facts". When FBPS are solved numerically, it is necessary to ensure that extraneous solutions are excluded. The trial-free-boundary method has been found to be a remarkably stable method for solving porous flow FBPS. We are aware of only one instance when the trial-free-boundary method was found to be unstable and this occurred when R. L. Taylor and Brown [1967] attacked a problem involving an overhanging interface with a downstream reservoir. We believe that the instability reported by Taylor and Brown is directly linked to the mathematical difficulties mentioned in remark (iii) above.

3. Seepage

There are two basic types of seepage problem: seepage from sources (reservoirs, ponds, channels, canals, infiltration ditches); seepage towards sinks (wells, drainage ditches, downstream reservoirs, tail waters). These problems can be combined in various ways, and the geometry can be further complicated by the introduction of impervious barriers, etc. We classify the problems as follows: flow through dams, flow from channels, flow to drainage ditches, flow towards wells. This classification is purely one of convenience, and several authors have developed computer programs which can handle more than one type of problem.

R. L. Taylor [1966] gives a general program for solving plane and axisymmetric porous flow problems with general geometry. Two applications of this program (flow through an inhomogeneous trapezoidal dam, and flow to a well) are given by R. L. Taylor and Brown [1967]. A later version of this program is given by Kealy and Busch [1971].

Neuman [1972] gives a general program for solving plane and axisymmetric saturated-unsaturated porous flow problems.

3.1. Seepage through dams

In many parts of the world, dams are constructed from earth or, less frequently, rock. Figures 1 and 2 illustrate some, but not all, of the various possibilities.

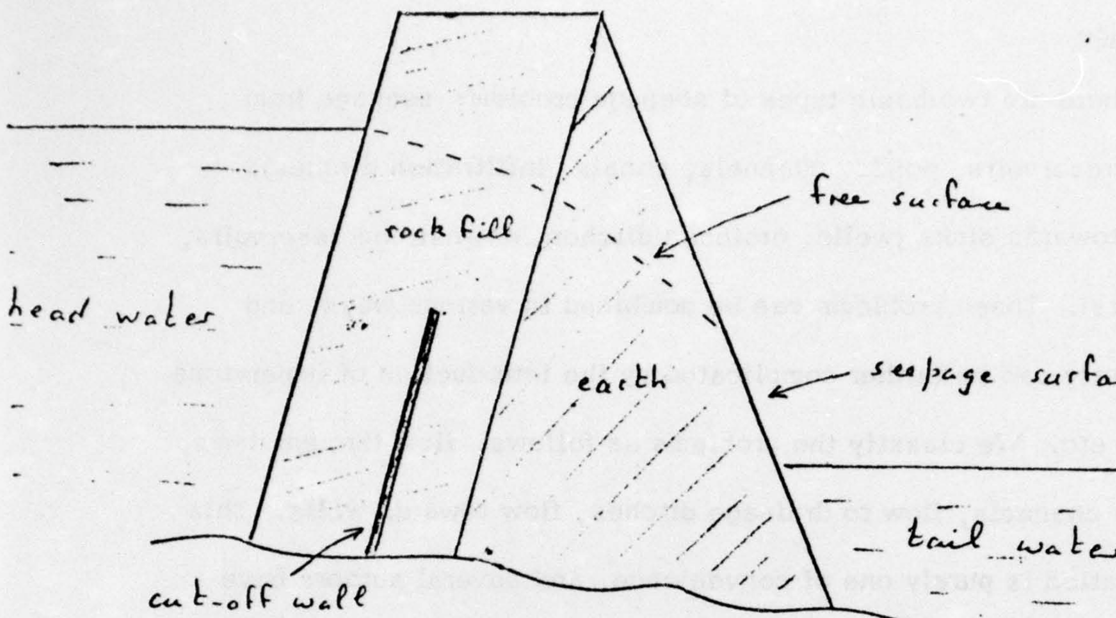


Figure 1: Seepage through a dam: example 1

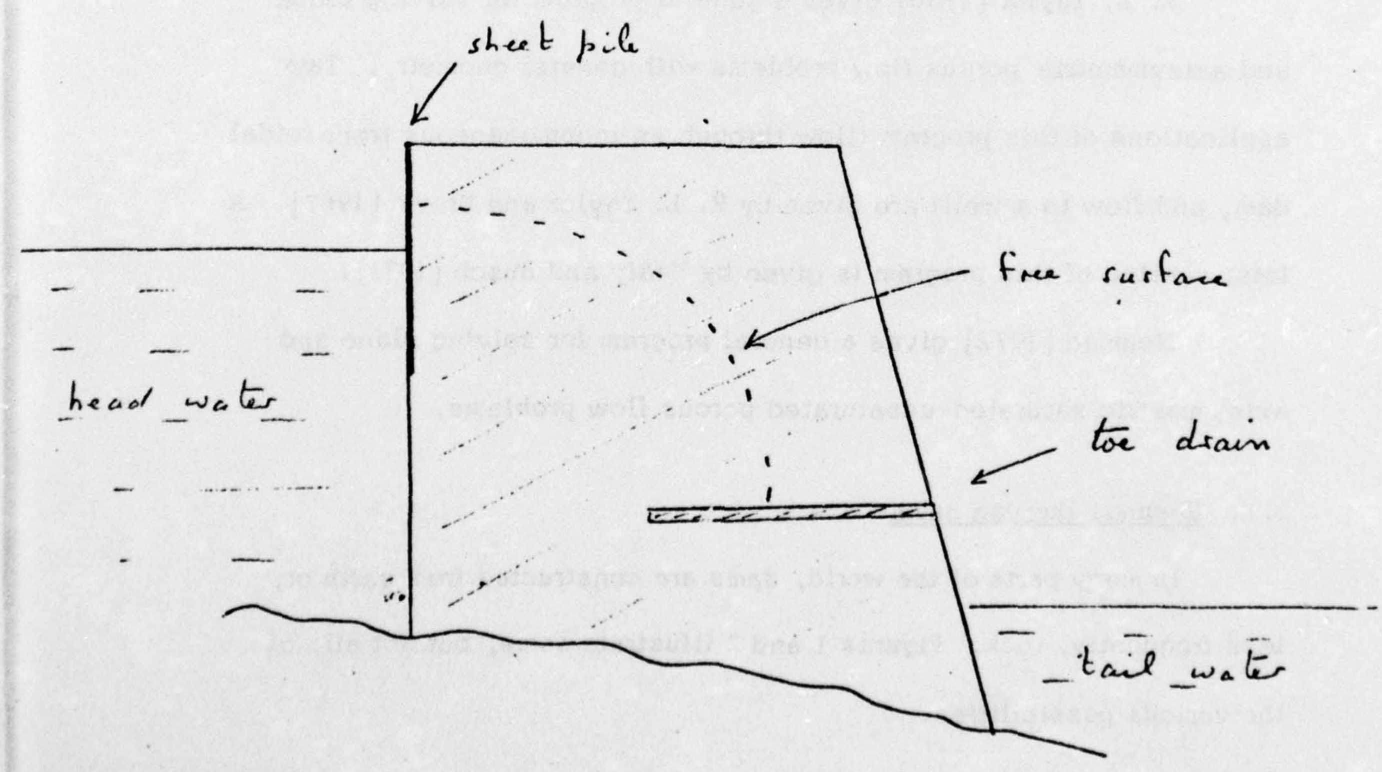


Figure 2: Seepage through a dam: example 2

The material in this section is subdivided as follows: simple rectangular dams, simple trapezoidal dams, polygonal dams, and other geometries.

3.1.1. Seepage through simple rectangular dams

The geometry is shown in Figure 1.

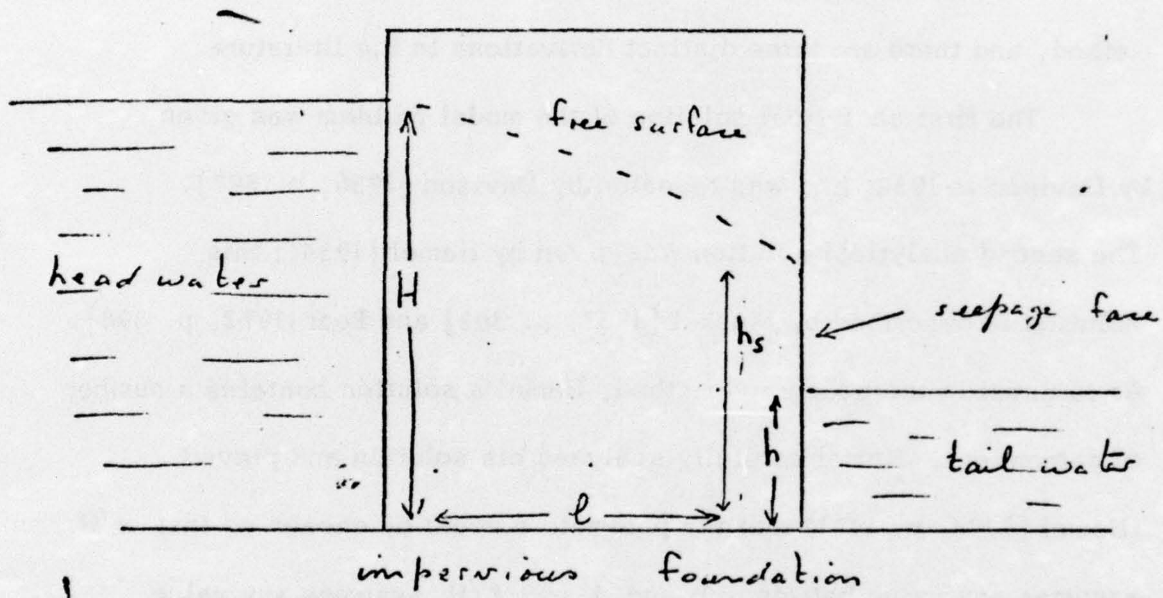


Figure 1: Seepage through a simple rectangular dam

The case of a homogeneous dam with constant pressure (atmospheric) on the FB has been considered by many authors and serves as a model problem for seepage through dams. This section is therefore divided into two subsections: the first subsection considers the model problem; and the second subsection considers generalizations.

3.1.1.1. Seepage through simple rectangular dams: the model problem

The geometry is shown in Figure 1 of section 3.1.1. In the model problem, which is considered here, it is assumed that the dam is homogeneous and that the pressure on the FB is constant (atmospheric).

The model problem can be solved analytically using the hodograph method, and there are three distinct derivations in the literature.

The first analytical solution of the model problem was given by Davison in 1932, and was repeated by Davison [1936, p. 897]. The second analytical solution was given by Hamel [1934]; this solution is described by Muskat [1937, p. 303] and Bear [1972, p. 398]. As is usual in the hodograph method, Hamel's solution contains a number of parameters. Hamel carefully analyzed his solution and proved (Hamel [1934, p. 147]) that the parameters could be chosen so that h/H assumes any value between 0 and 1 and ℓ/H assumes any value less than ∞. In other words, Hamel proved that a solution exists for all $H > 0$, $h < H$, $\ell > 0$.

Hamel and Günther [1935] and Muskat [1935] have evaluated Hamel's solution for several values of the parameters. This work is discussed by Muskat [1937, p. 309] and the results are quoted in Table 1 below.

The third analytical solution of the model problem is given by Polubarinova-Kochira [1962, p. 284]. This solution gives the geometrical dimensions (Polubarinova-Kochina [1962, p. 289]),

$$\ell = C \int_0^{\pi/2} \frac{\hat{K}[\alpha+(\beta-\alpha)\sin^2\Psi]}{\sqrt{1-\alpha-(\beta-\alpha)\sin^2\Psi}} d\Psi , \quad (1)$$

$$H = C \int_0^{\pi/2} \frac{\hat{K}[\beta+(1-\beta)\sin^2\Psi]\, d\Psi}{\sqrt{\beta-\alpha+(1-\beta)\sin^2\Psi}} , \quad (2)$$

$$h = C\sqrt{\alpha} \int_0^{\pi/2} \frac{\hat{K}(\alpha \sin^2\Psi)\sin\Psi\, d\Psi}{\sqrt{(1-\alpha\sin^2\Psi)(\beta-\alpha\sin^2\Psi)}} . \quad (3)$$

in terms of parameters α, β, C. Here, $\hat{K}$ denotes the complete elliptic integral of the first kind,

$$\hat{K}(\xi) = K(\sqrt{\xi}) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-\xi\sin^2\varphi}} . \quad (4)$$

The height of the seepage surface is (Polubarinova - Kochina [1962, p. 289]),

$$h_s = h + C\int_0^{\pi/2} \frac{\hat{K}(\cos^2\Psi)\sin\Psi\cos\Psi\, d\Psi}{\sqrt{(1-\alpha_1\sin^2\Psi)(1-\beta_1\sin^2\Psi)}} \quad (5)$$

where $\alpha_1 = 1 - \alpha$, $\beta_1 = 1 - \beta$. The FB is given in parametric form (Polubarinova - Kochina [1962, p. 290]),

$$x = \ell - C\int_0^{\Psi} \frac{\hat{K}(\sin^2\Psi)\sin\Psi\, d\Psi}{\sqrt{(1-\alpha\sin^2\Psi)(1-\beta\sin^2\Psi)}} \quad (6)$$

$$y = h+h_s + C\int_0^{\Psi} \frac{\hat{K}(\cos^2\Psi)\sin\Psi\, d\Psi}{\sqrt{(1-\alpha\sin^2\Psi)(1-\beta\sin^2\Psi)}} \quad (7)$$

for $0 \leq \Psi \leq \pi/2$. Polubarinova - Kochina [1962, p. 290] evaluates these formulas for various values of the parameters α, β, C.

There are two misprints in Polubarinova-Kochina [1962, p. 289]: the term in the denominator of (2) is given as $(\beta-\alpha) - (1-\beta) \sin^2 \Psi$; the range of integration of (5) is given as $[0, 1]$. Also, in the integrals for ℓ, H, and h given by Polubarinova-Kochina [1962, p. 288], the term ζ in the denominator should be $\sqrt{\zeta}$.

In addition to the three analytical solutions described above, the model problem has been solved approximately by many authors. The results are summarized in Table 1 below.

Author (Method)	H	ℓ	h	h_s
Hamel and Günther [1935] (Evaluation of Hamel's solution)	.321 ± .009	.1585 ± .002	.081 ± .005	.205 ± .007
Muskat [1935] (see Muskat [1937, p. 314]) (Evaluation of Hamel's solution)	.3223 .6695 .67204 .8715	.1619 .4439 .3293 .4843	.0842 .1579 0 0	.2067 .3598 .4300 .5192
Polubarinova-Kochina [1962, p. 292] (Evaluation of solution)	Many results plotted graphically			
Breitenöder [1942, p. 56] (Graphical methods in hodograph plane)	1.49	1.08	0	.74
Wyckoff and Reed [1935, p. 398] (trial free ary method; analog)	.322 .68*	.157 .61*	.083 0	.206 .29*
Shaw and Southwell [1941, p. 9] (see Southwell [1946, p. 207]) (trial free boundary method; finite differences)	24	16	4	12.75*
McNown, Hsu, and Yih [1955, p. 668] (trial free boundary method; finite differences)	1 (24)	2/3 (16)	0 (0)	.43 (10.32)

	H	ℓ	h	h_s	Remarks
Finnemore and Perry [1968, p. 1066] (trial free boundary method; finite differences)	24	16	4	12.268* 12.192* 12.213*	Different initial guesses
Herbert and Rushton [1966] (trial free boundary method; resistance network)	32	16	4	20.8*	
J.M. Taylor [1971; p. 56, p. 57, p, 65] and Aitchison [1972] (trial free boundary method; finite differences)	1	2/3	0	.5268	(Δx=H/24)
	(24)	(16)	(0)	(12.64)	
	1	2/3	1/6	.5330	(Δx=H/24)
	(24)	(16)	(4)	(12.79)	
	1	2/3	1/6	.5314	(Δx=H/48)
	(24)	(16)	(4)	(12.75)	
Comincioli, Guerri, and Volpi [1971] (trial free boundary method; finite differences)	.322	.162	.084	.20611 .20684	(Method M_1) (Method M_2)
	24	16	4	12.892 12.368	(Method M_1) (Method M_2)
Szabo and McCaig [1968] (time-dependent; finite differences	19.7	26.56	1.3	5.3*	See France et al. [1971]
Herbert [1968] (Time-dependent; resistance network)	16	16	0	7.3*	
France, Parekh, Peters, and Taylor [1971] (time-dependent; finite elements)	19.7	26.56	1.3	5.3*	
	16	16	0	6.8*	
Comincioli, Guerri, and Volpi [1971] (variational inequalities; finite differences)	.322	.162	.084	.2364 .2043 .2150	(Δx=H/15) (Δx=H/30) (Δx=H/60)
	24	16	4	14.4	(Δx=H/30)
Kealy and Busch [1971, p.9] (trial-free-boundary method; finite elements) (Also Kealy and Williams [1971, p. 145])	6	5	1	2.6*	

Table 1: Numerical solutions for the model problem (see text for explanatory comments)

A number of explanatory comments must be made regarding Table 1:

(i) The values marked by an asterisk have been estimated by us from graphs.

(ii) The values in brackets have been scaled by us from the results in the previous line and rounded to four significant figures for easy comparison.

(iii) The results obtained using variational inequalities are not strictly comparable to the other results because a different formulation is used. The value of h_s shown in Table 1 is obtained using the first y-coordinate for which the solution vanishes on the y grid-line adjacent to the downstream face of the dam. The results given here were obtained using a modified version of the program of Comincioli, Guerri, and Volpi [1971].

In addition to the results shown in Table 1, both J. M. Taylor [1971] and Muskat [1935] give the solutions for other values of the parameters and unpublished solutions have been obtained by Symm [1975] (trial free boundary method with integral equations) and Ferriss [1972] (brute force transformation to a rectangular domain).

The evaluation of the solutions of Hamel and Polubarinova-Kochina is a difficult task by hand but relatively simple on a computer. It is therefore unfortunate that this work has been overlooked and that many authors have computed numerical solutions for the model problem without comparing their results with the exact solutions. In Table 1 several problems occur frequently. For comparison we give in Table 2 the corresponding values obtained by us by evaluating formulas (1) to (5). The integrals were evaluated using a double-precision version of the CADRE subroutine of de Boor [1971]. The values of α and β were found using a single-precision version of the method of Brown [1969] applied to the equations

$$\frac{\ell}{\ell_r} - \frac{H}{H_r} = 0, \qquad h_r \frac{\ell}{\ell_r} - h = 0, \tag{8}$$

where ℓ, H, and h are given by equations (1) to (3) and where ℓ_r, H_r, and h_r denote the desired values of ℓ, H, and h, respectively. The solutions given in Table 2 are believed to be correct to the number of decimals shown. We are grateful to Julia Gray for her help with the programming.

H	ℓ	h	h_s	α	β
24	16	4	12.7132	.095126	.465367
24	16	0	12.5674	0	.416089
.322	.162	.084	.204460	.098554	.199823

Table 2: Analytical evaluation for the model problem

We now consider the analysis of the model problem.

It was found by Charni [1951] that the flow Q through the dam is given exactly by the formula

$$Q = k(H^2 - h^2)/2\ell , \tag{9}$$

where k denotes the permeability. Formula (9) is derived by Polubarinova - Kochina [1962, p. 282] and Bear [1972, p. 367]. Formula (9) is also obtained by using the Dupuit - Forchheimer approximation (see Bear [1972, p. 366]). Many authors have solved the model problem numerically, computed the flow Q, and remarked how accurate the Dupuit - Forchheimer approximation (9) is, not realizing that is in fact exact.

There are several proofs of existence and uniqueness. Hamel [1934] proved existence by analysing in detail the analytic solution obtained using the hodograph method. Both Hamel [1934, p. 147] and Davison [1936, p. 882] indicated that they believed that the solution was unique, but they were unable to prove this. Miranda [1969] proves existence and uniqueness using conformal mapping techniques. Baiocchi [1971] showed that the problem could be reformulated as a variational inequality, and proved existence and uniqueness; for a more detailed description of this work see Baiocchi [1972], Baiocchi, Comincioli, Magenes, and Pozzi [1973].

It is also possible to prove several results about the FB. Davison [1936a] used the maximum principle to prove a number of qualitative properties of the FB. Shaw and Southwell [1941, p. 15] use the maximum principle to prove that the FB is monotonically decreasing. Miranda [1969, p. 76], Baiocchi [1972, p. 124] and Baiocchi, Comincioli, Magenes, and Pozzi [1973, p. 32] prove that the FB is analytic. J. M. Taylor [1971] (see also Aitchison [1972]) obtains an expansion for the FB in the neighbourhood of the seepage face; this is a special case of the results of Polubarinova-Kochina [1962, p. 276] and Kravtchenko, de Saint Marc, and Boreli [1955].

3.1.1.2. Seepage through simple rectangular dams: the general case

The geometry is shown in Figure 1 of section 3.1.1.

The following generalizations of the model problem have been considered:

Evaporation

Davison and Rosenhead [1940, p. 352] derive an analytic solution for the case when there is no seepage face. Pozzi [1974, p. 14; 1974a, p. 16] derives an analytic solution in three special cases, for all of which there is no seepage face.

Pozzi [1974] reformulates the problem as a variational inequality and proves existence and uniqueness theorems. Pozzi [1974a, p. 6] proves that the FB is analytic. Pozzi [1974a] also gives numerical results obtained by solving the variational inequality using finite differences.

Infiltration

Pozzi [1974a, p. 24] obtains some non-existence results which lead him to question the correctness of the problem.

Variable permeability

Outmans [1964] obtains the flow Q for horizontally or vertically stratified dams.

Baiocchi, Comincioli, Magenes, and Pozzi [1973] prove existence and uniqueness for horizontally or vertically stratified dams.

Benci [1973, 1974] uses variational inequalities to prove existence and uniqueness for the case

$$k = \exp(f(x) + g(y)).$$

3.1.2. Seepage through simple trapezoidal dams

The geometry is shown in Figure 1 in which α, β, H, h, and either ℓ or ℓ_C must be specified. Unless otherwise indicated we consider the case of a homogeneous isotropic material.

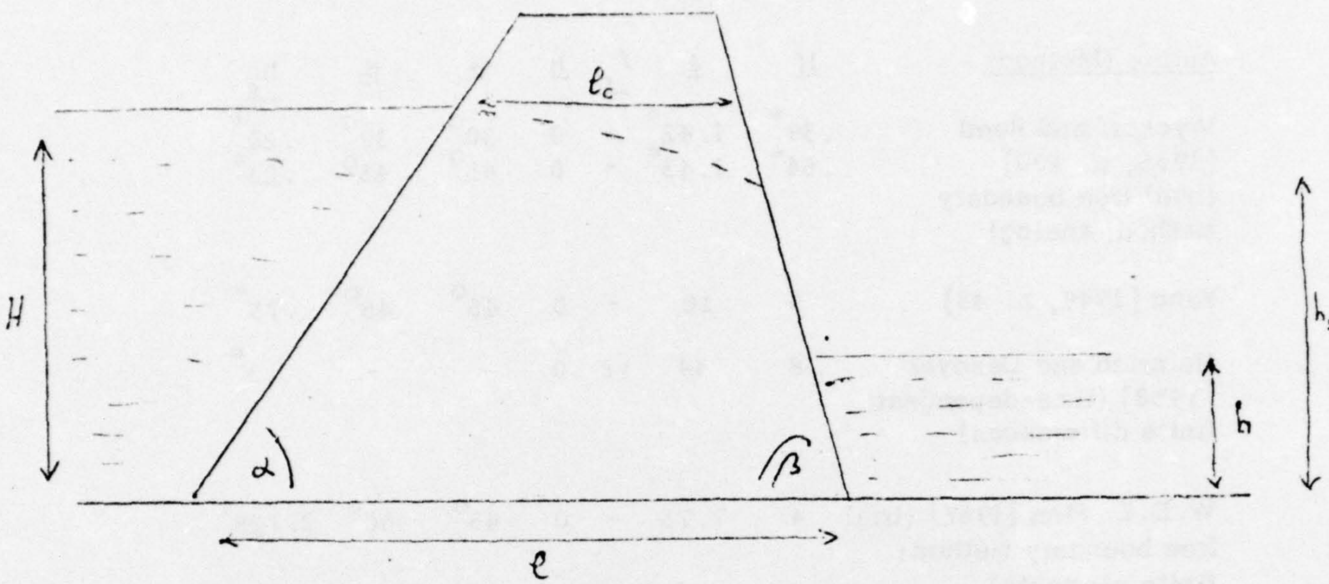


Figure 1: Seepage through a simple trapezoidal dam

The case of a trapezoidal dam is more complicated mathematically than the case of a horizontal rectangular dam because the discharge Q is not known.

For the case $\beta = 90^\circ$ Baiocchi, Comincioli, Magenes, and Pozzi [1973, p. 73] prove that there exists at most one solution, and Comincioli [1974] proves that there exists a solution and also describes a convergent numerical method. So far as we know, the existence of solutions for $\beta \neq 90^\circ$ is not known.

Numerical solutions have been obtained in the following cases:

Author (Method)	H	ℓ	ℓ_c	h	α	β	h_s
Wyckoff and Reed [1935, p. 400] (trial free boundary method; analog)	.39* .54*	1.42* 1.43*	- -	0 0	30° 45°	30° 45°	.26* .25*
Yang [1949, p. 48]	3	10	-	0	45°	45°	.75*
Heinrich and Desoyer [1958] (time-dependent; finite differences)	8	44	12	0	-	-	3*
W.D.L. Finn [1967] (trial free boundary method; finite elements)	4	7.75	-	0	45°	60°	2.125*
France, Parekh, Peters, and Taylor [1971, p. 177] (time-dependent; finite elements)	12	42	-	0	$\tan^{-1}(\frac{12}{5})$	$\tan^{-1}(\frac{12}{19})$	2.4*
Fenton [1972a] and Parkin [1971] (trial free boundary method; finite elements)	1 1	- -	.5 .5	.5 .5	33° 45°	33° 45°	.75* .75*
Baiocchi, Comincioli, Guerri, and Volpi [1973, p. 57] (variational inequalities; finite differences)	1	3	-	2	0	45°	90°
Comincioli [1974b] (variational and quasi-variational inequalities; finite differences)		various				90°	

*Estimated by us from graphs.

Table 1. Numerical solutions for a simple trapezoidal dam

Fenton [1972a] lists his program and gives results for many values of the parameters. The work of Fenton [1972a] and Parkin [1971] is for the general nonlinear law $F(t) = at^{m-1}$ (see section 1.3.2), but, for consistency, the results quoted above are for the linear case $m = 1$.

Freeze [1971a] uses a time-dependent method with finite differences to solve several problems of flow through non-homogeneous trapezoidal dams. Freeze considers saturated-unsaturated flow, but he compares his results with those for saturated flow.

Pettibone and Kealy [1971] use the trial free boundary method with finite elements for flow through an inhomogeneous trapezoidal dam (a mill-tailing dam).

In addition, Weinig and Shields [1936], A. Casagrande [1940, p. 313], and Neuman and Witherspoon [1970, p. 895] have obtained approximate solutions, but the numerical values of the geometrical parameters are not given.

3.1.3. Seepage through polygonal dams

There is an immense number of possibilities. The following problems have been considered:

Cut-off walls

A cut-off wall is a wall (usually sloping and impervious) built to cut off the flow. Volker [1969] has used the trial free boundary method

with finite elements to solve the problem of nonlinear flow through a polygonal dam with cut-off wall (Figure 1).

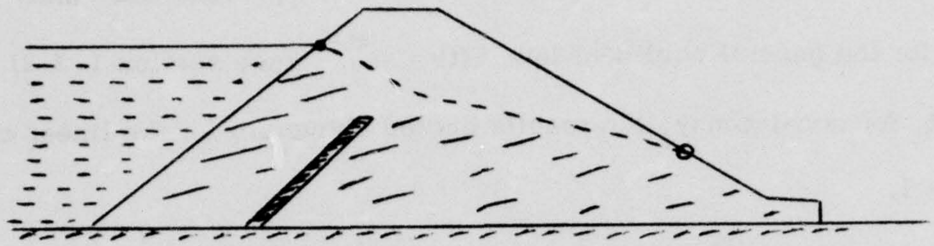


Figure 1. Flow with impervious cut-off wall
(based on Volker [1969, p. 2109])

Toe Drains

Harr [1962, p. 223] gives the analytical solution for a trapezoidal dam with toe drain (Figure 2).

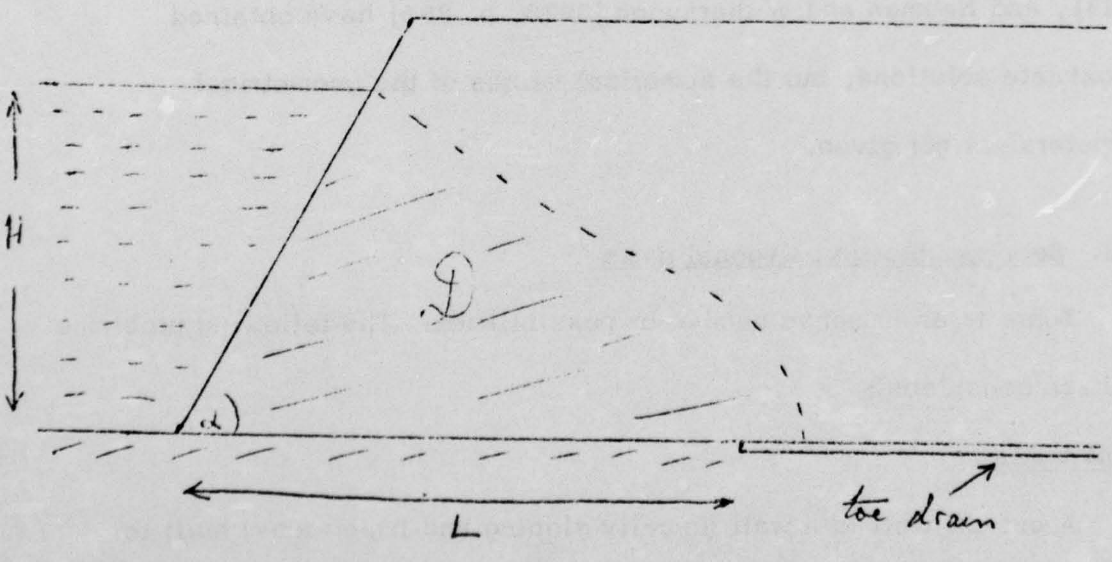
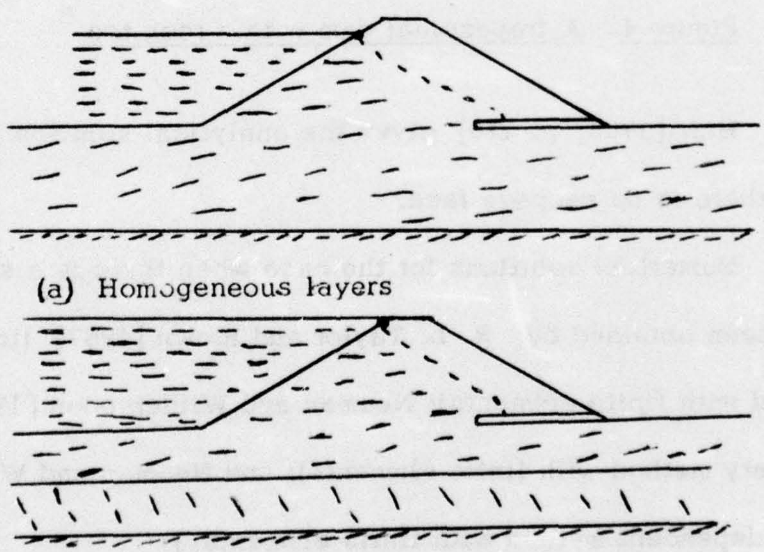


Figure 2. Trapezoidal dam with toe drain

Jeppson [1966, p. 12; 1968] has used finite differences in the $\phi\psi$-plane to solve the problem of Figure 2 for several values of H, L, and α. Jeppson compares his results with the graphical results of A. Casagrande [1940] and the analytical results.

Shaw and Southwell [1941] used the trial free boundary method with finite differences to solve two problems of a trapezoidal dam with toe drain on a porous layer (Figure 3). In the first problem the material was isotropic and homogeneous (Figure 3a); on the second problem there was a sublayer of different permeability. (Figure 3b). Neuman and Witherspoon [1970, p. 896] use the trial-free-boundary method with finite elements to solve a similar problem.



(a) Homogeneous layers

(b) Two layers

Figure 3: Trapezoidal dams with toe drains on pervious layers (based on Shaw and Southwell [1941, p. 4])

Trapezoidal dams with rock toes

A trapezoidal dam with rock toe is shown in Figure 4.

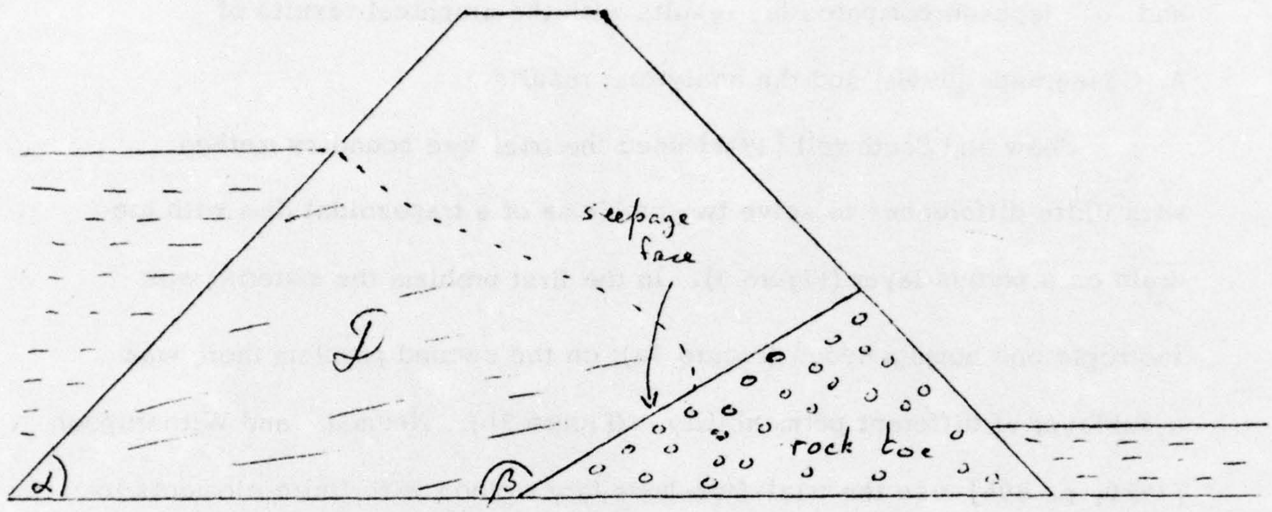


Figure 4. A trapezoidal dam with a rock toe

Harr [1962, p. 210] gives the analytical solution for the case when there is no seepage face.

Numerical solutions for the case when there is a seepage face have been obtained by: R. L. Taylor and Brown [1967] (trial free boundary method with finite elements); Neuman and Witherspoon [1970] (trial free boundary method with finite elements); and Neuman and Witherspoon [1971a] (time-dependent method with finite elements.)

Mill-tailing dams

Mill-tailing dams are dams which are built to contain industrial effluents. Kealy and Soderberg [1969] give a general survey of the design

problems associated with mill-tailing dams. The most important aspect is the stability of the dams. Instability can lead to disasters such as at Aberfan in Wales in which a school was buried. The location of the FB substantially affects the stability characteristics of a dam (Kealy and Williams [1971, p. 153].

Kealy and Busch [1971] (see also Kealy and Williams [1971, p. 152]) use the trial free boundary method with finite elements to solve anisotropic non-homogeneous mill-tailing dam problems (Figure 5).

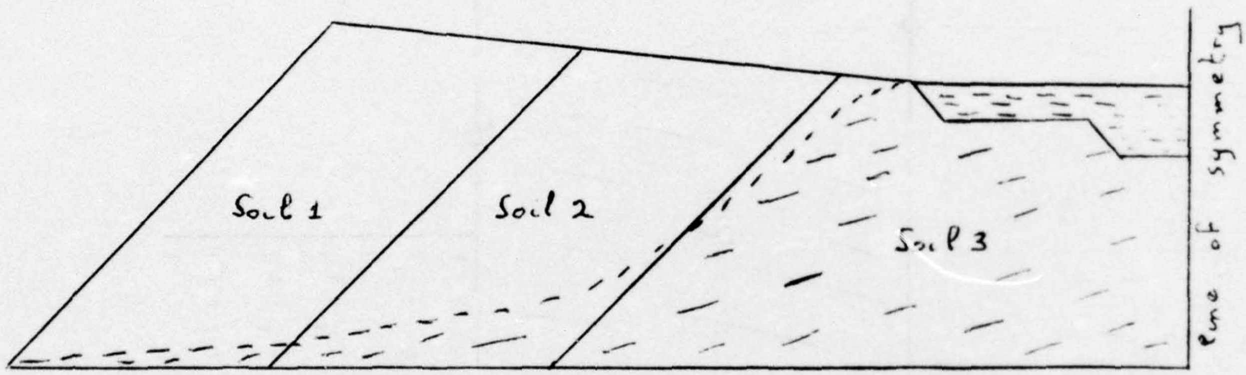


Figure 5. A typical mill-tailing dam (based on Kealy and Busch [1971, p. 16]).

An interesting feature of these problem is that sometimes the FB is concave (as in Figure 5). This is in agreement with observations (Kealy and Busch [1971, p. 6 and p. 25).

Sheetpiles

The problem of a dam with sheetpiles introduces interesting additional mathematical and numerical difficulties because the upstream point of detachment is free. (See Figure 6).

Baiocchi, Comincioli, Magenes, and Pozzi [1973, p. 46] prove existence and uniqueness for a rectangular dam (Figure 6). So far as we are aware, there are no numerical solutions.

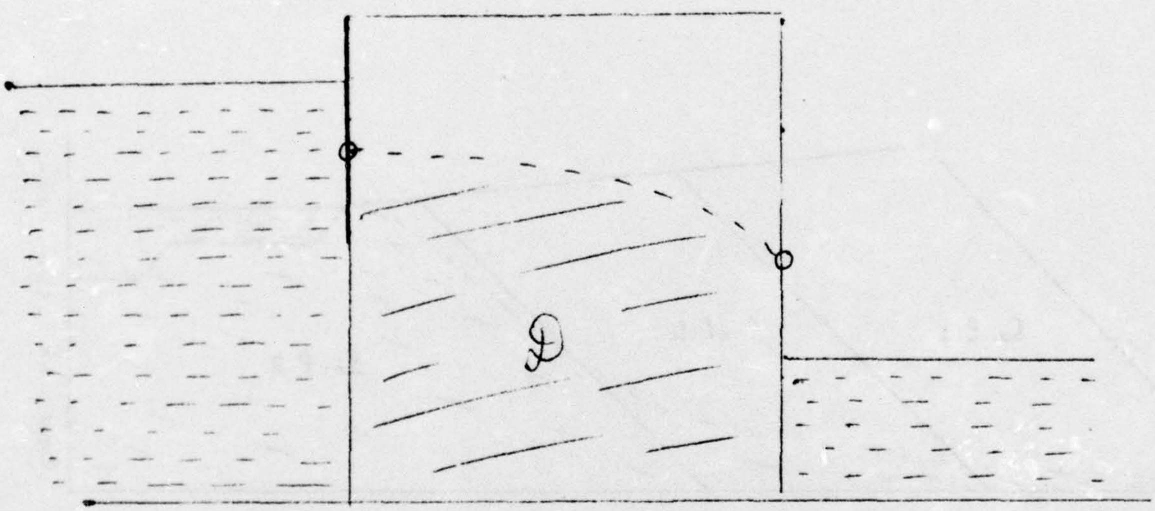


Figure 6. Rectangular dam with sheetpile

Harr [1962, p. 176] gives the analytic solution for a dam with sheetpile wall and toe drain (Figure 7).

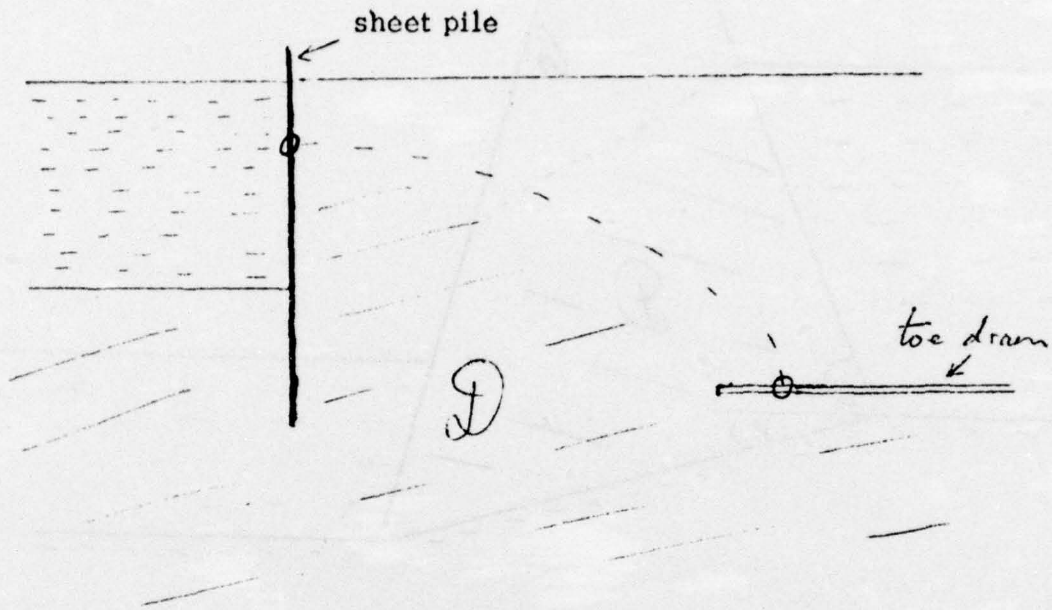


Figure 7. Dam with sheet pile and toe drain

W. D. L. Finn [1967, p. 46] uses the trial free boundary method with finite elements to solve the problem of a trapezoidal dam with two sheetpiles. Maione and Franzetti [1969] give experimental results for a particular dam, the Kainji dam in Niger.

Trapezoidal dams with sloping bases

The geometry is shown in Figure 8.

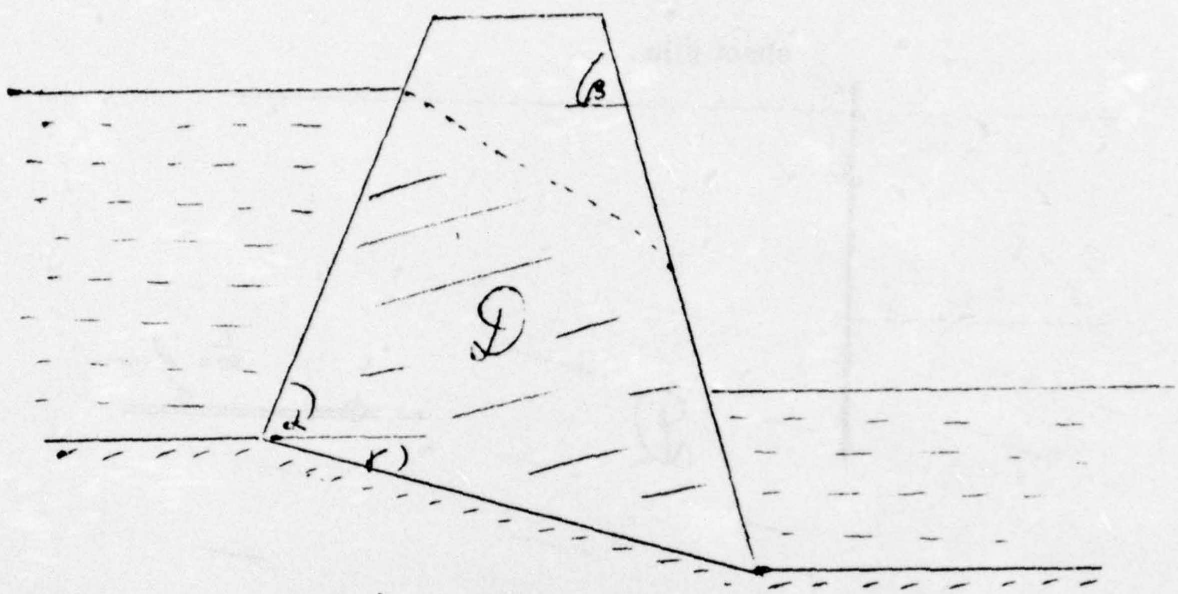


Figure 8. Trapezoidal dam on a sloping base

Baiocchi, Comincioli, Magenes and Pozzi [1973, p. 76] use variational inequalities to prove uniqueness for the case $\beta = \pi/2$, and Baiocchi, Comincioli, Guerri, and Volpi [1973, p. 65] give numerical results for the case $\alpha = \beta = \pi/2$.

For the case $\beta = \pi/2$, Comincioli [1974a] proves existence and uniqueness using a modification of the method of variational inequalities. Comincioli also gives some numerical results.

Harr [1962, p. 208] gives an exact analytic solution for the case $\beta = \pi$.

General polygonal dams

Davison [1936a] considers the flow through general polygonal dams [Figure 9).

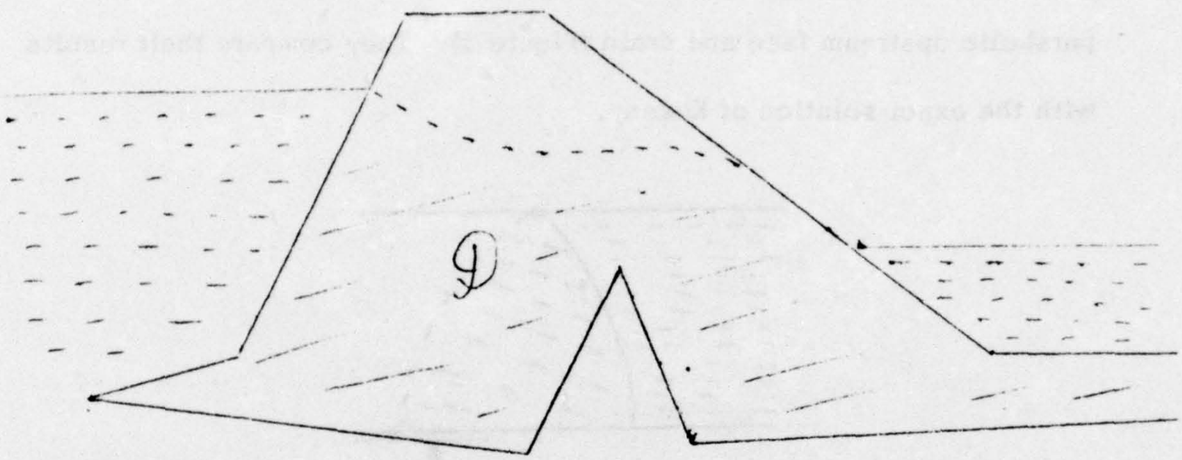


Figure 9: A general polygonal dam

Three-dimensional dams

Freeze [1971a] uses the time-dependent method with finite differences to solve a problem of flow through a three-dimensional polyhedron.

Three-dimensional problems have frequently been solved using electrolytic tanks (Bear [1972, p. 702], Polubarinova-Kochina [1962, p. 463].

3.1.4. Seepage through dams with general geometry

The following problems have been considered:

Herbert and Rushton [1966, p. 68] use the trial free boundary method with resistance networks for the seepage through a dam with parabolic upstream face and drain (Figure 1). They compare their results with the exact solution of Kozeny.

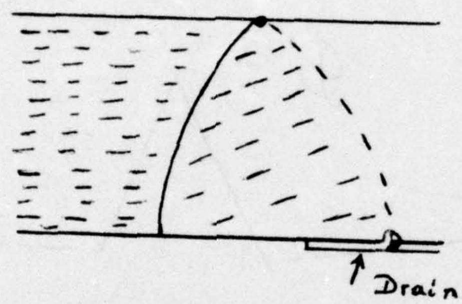


Figure 1: Dam with parabolic upstream face and drain (based on Herbert and Rushton [1966, p. 68])

Volker [1969] uses the trial free boundary method with finite elements to solve the problem of nonlinear flow through a "trapezoidal" dam with a slightly curved downstream face.

Baiocchi [1974] has shown how the problem of flow through a dam of general cross-section (Figure 2) can be reformulated as a quasi-variational inequality. Comincioli [1974b] applies this method to obtain numerical solutions; Comincioli gives numerical results for a trapezoidal dam but points out that the method can be applied to a wide range of problems.

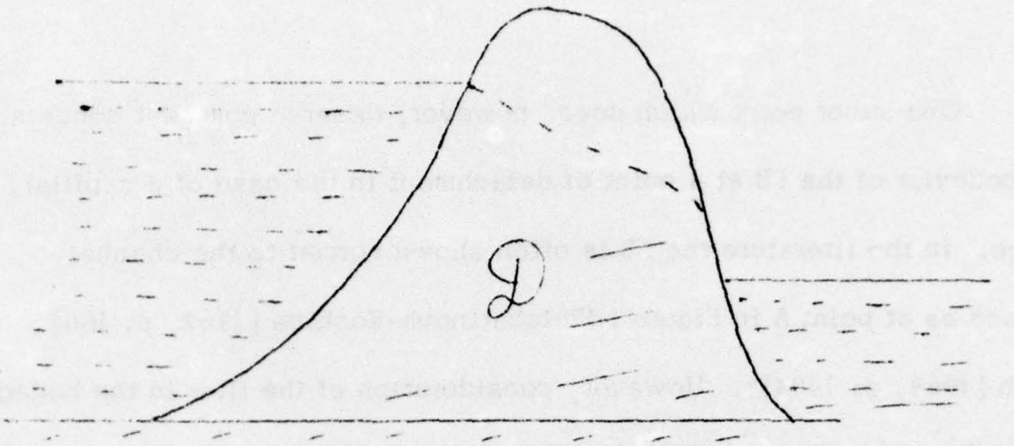


Figure 2: A dam with general cross-section

3.2. Seepage from channels

There is no standard terminology, and channels are also called infiltration ditches, canals, streams, or ponds (axisymmetric).

It is worth mentioning (Garg and Chawla [1970, p. 1261]) that the analysis of seepage losses from channels is an important problem. In some irrigation systems in India so much water is lost from the channels bringing water to the fields that less than half the water originally available is delivered to the fields. Not only is that inefficient, but the ground alongside the channels becomes water-logged which creates additional difficulties. Hagan, Haise, and Edminster [1967, chapter 1] give a very interesting account of the irrigation systems of ancient civilizations.

Harr [1962, chapter 9] and Polubarinova-Kochina [1962, chapter 5] give useful summaries of the available analytical solutions.

One minor point which does, however, deserve comment concerns the behavior of the FB at a point of detachment in the case of a capillary fringe. In the literature the FB is often shown normal to the channel surface as at point A in Figure 1 (Polubarinova-Kochina [1962, p. 160], Bruch [1969, p. 1404]). However, consideration of the flow in the hodograph plane (Bruch [1969, p. 1405], Bear [1972, p. 285]) shows that the FB is horizontal as at point B in Figure 1.

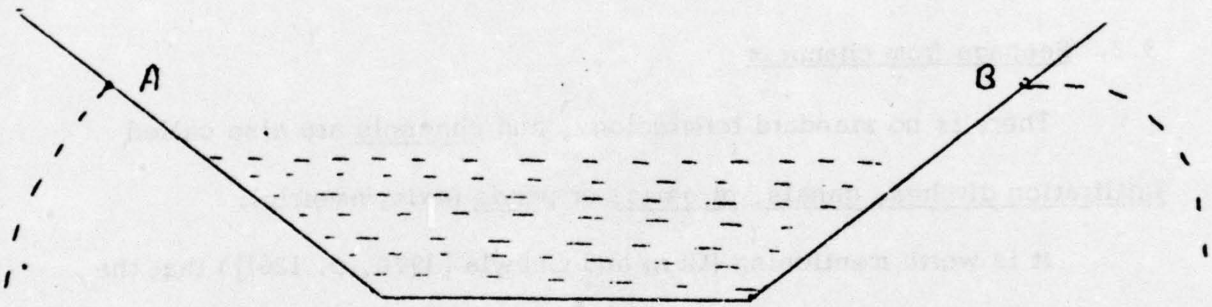


Figure 1: Behavior of the FB for a capillary fringe

3.2.1. Single channel into half-plane

The geometry is shown in Figure 1; water seeps from a channel into the surrounding soil.

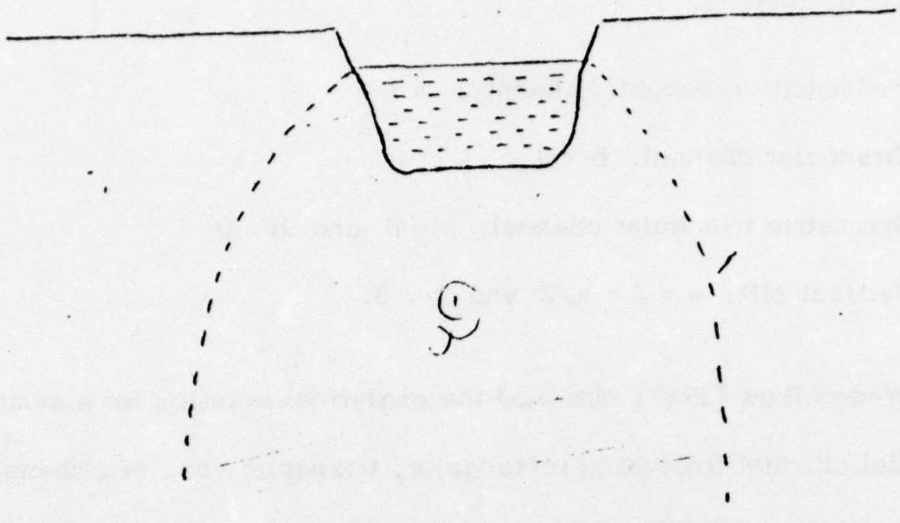


Figure 1: Seepage from a single channel into a half-plane

The case which has been considered most frequently is the case of a channel of trapezoidal cross-section (Figure 2).

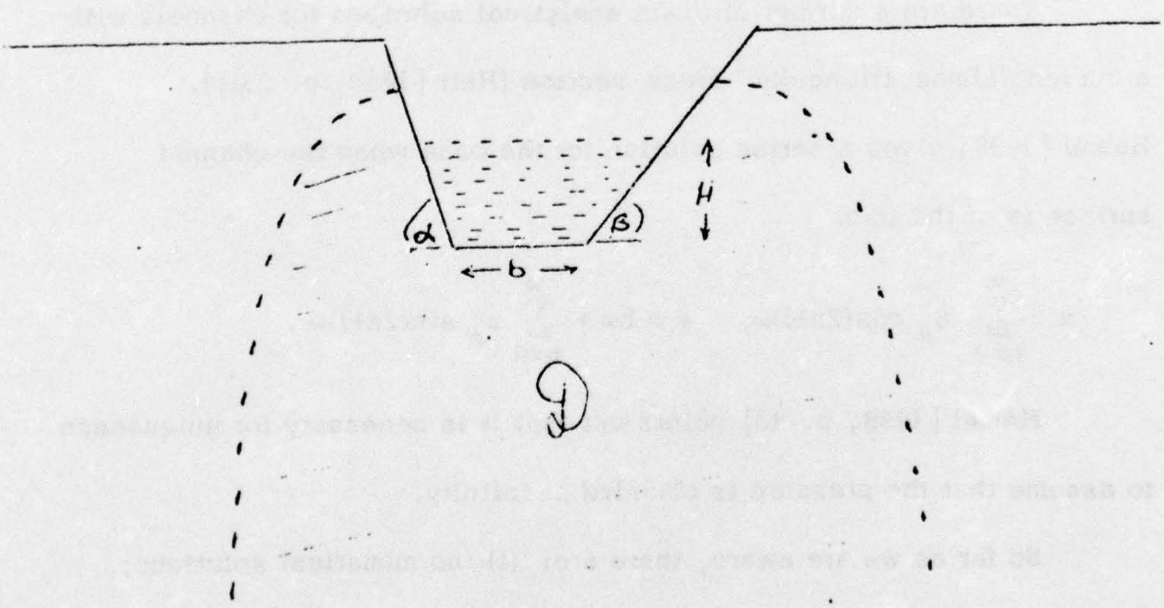


Figure 2: Seepage from a plane trapezoidal channel into a half-plane (with capillary fringe).

Special cases include:

(i) Symmetric trapezoidal channel: $\alpha = \beta$

(ii) Triangular channel: $b = 0$

(iii) Symmetric triangular channel: $\alpha = \beta$ and $b = 0$

(iv) Vertical slit: $\alpha = \beta = \pi/2$ and $b = 0$.

Wedernikow [1937] obtained the analytical solution for a symmetric trapezoidal channel (including rectangular, triangular, and slit channels as special cases); see Harr [1962, chapter 9], Polubarinova-Kochina [1962, page 149]. Later, Wedernikow [1940] obtained the analytical solution for a symmetric trapezoidal channel in the case of a capillary fringe; see Polubarinova-Kochina [1962, p. 160].

There are a number of exact analytical solutions for channels with a curved "almost triangular" cross-section (Harr [1962, p. 231]). Hamel [1938] gives a series solution for the case when the channel surface is of the form

$$x = \sum_{n=0}^{\infty} a_n \cos(2n+1)\omega, \qquad y = b\omega + \sum_{n=0}^{\infty} a_n \sin(2n+1)\omega .$$

Hamel [1938, p. 43] points out that it is necessary for uniqueness to assume that the pressure is bounded at infinity.

So far as we are aware, there are: (i) no numerical solutions; (ii) no analytic solutions for plane unsymmetric trapezoidal channels; (iii) no analytic solutions for axisymmetric problems. There are of course numerical solutions for flow into a finite layer (see section 3.2.2).

3.2.2. Single channel into finite layer

The geometry is shown in Figure 1 for a trapezoidal channel: water seeps from the channel into the surrounding soil of finite depth; at depth D there is a drain or pervious layer.

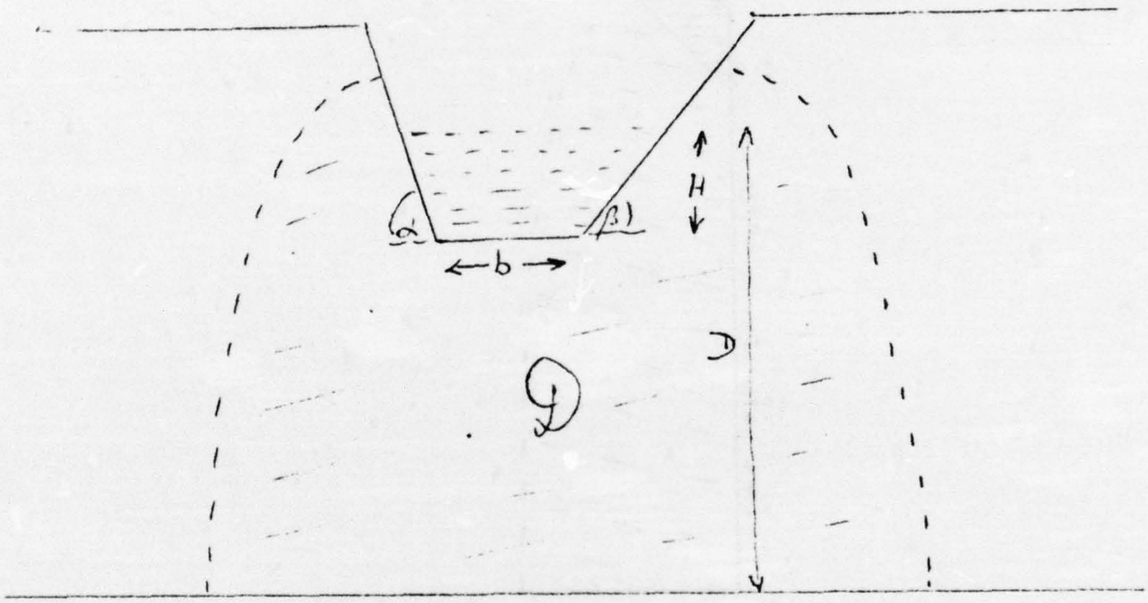


Figure 1: Seepage from channel into finite layer

We consider the plane and axially symmetric cases separately.

The case of a plane symmetric triangular channel has been considered by several workers. Bruch and Street [1967] give an analytical solution and find that it compares well with experimental results for $\alpha = 45^{\circ}$, $H = 4$, $D = 46$; Bruch and Sainz [1972] use an interactive computer terminal to evaluate this solution. Jeppson [1968a] uses finite differences in the $\phi\psi$-plane and compares his results with

those of Bruch and Street [1967] for the case $\alpha = 45^{\circ}$, $H = 9$, $D = 51$; Bruch and Sainz [1972] plot their results (see Figure 2) and compare them with those of Jeppson (not shown). Jeppson [1968a] also gives results for several other values of the parameters.

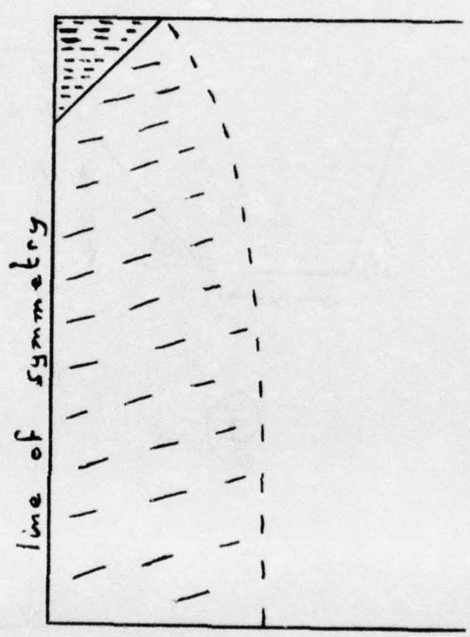


Figure 2: Seepage from a triangular channel for $\alpha = 45^{\circ}$, $H = 9$, $D = 51$ (based on Bruch and Sainz [1972, p. 522]).

Plane symmetric trapezoidal channel problems have been solved for a variety of governing equations by Jeppson using finite differences in the $\phi\psi$-plane:

Reference	Governing Equations
Jeppson [1968a]	Homogeneous, isotropic.
Jeppson [1969a]	Isotropic, $k = Ax + By + C$.
Jeppson [1968b]	Two soil layers 1 and 2 with constant permeability tensors. $\underline{K}^1 = \begin{pmatrix} k_x^{(1)} & 0 \\ 0 & k_y^{(1)} \end{pmatrix}$, $\underline{K}^2 = \begin{pmatrix} k_x^{(2)} & 0 \\ 0 & k_y^{(2)} \end{pmatrix}$. If the interface between the layers is not horizontal then we must have $k_x^{(1)}/k_y^{(1)} = k_x^{(2)}/k_y^{(2)}$.
Jeppson and Nelson [1970]	Partially saturated flow with $k = \begin{cases} k_0/(p/p_0)^{\alpha} + b, & \text{if } p < 0, \\ k_0/b, & \text{if } p \geq 0. \end{cases}$

Jeppson [1968c] has considered axisymmetric problems. Jeppson uses finite differences in the axially symmetric $\phi\psi$-plane and allows the permeability to depend linearly upon the depth y. The channels are not specified but appear as part of the solution. In the problems solved the channels are approximately trapezoidal, and Jeppson [1968c, p. 1279] asserts that the solution for a channel of prescribed shape could be obtained by an iterative procedure. Neuman and Witherspoon [1970] use the trial free boundary method with finite elements to solve the same problem and compare their results with those of Jeppson.

3.2.3. Seepage from multiple channels

Bruch [1969] has applied the hodograph method to the problem of a periodic array of parallel (plane) symmetric triangular channels with a capillary fringe underlain by a drainage layer at finite depth (Figure 1). We are not aware of any numerical solutions.

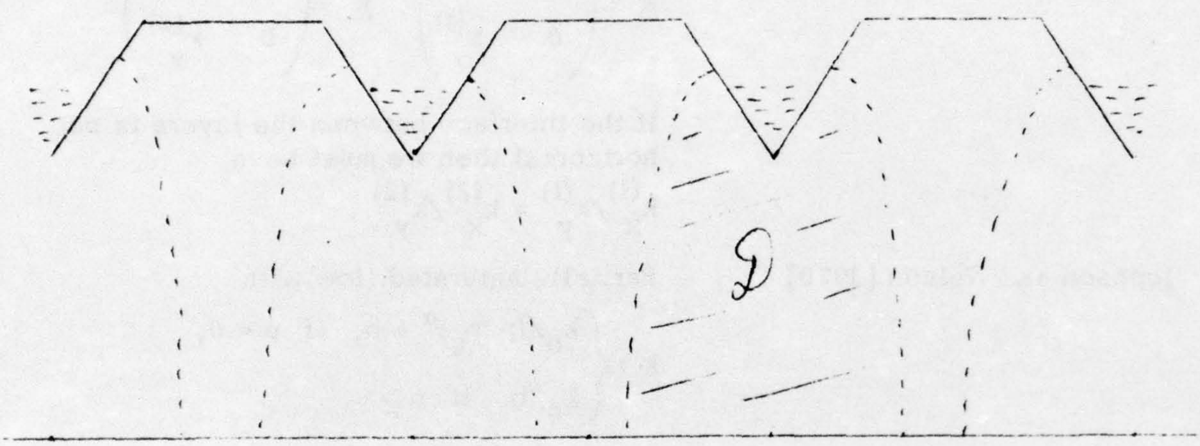


Figure 1: Seepage from periodic triangular channels

3.3. Drainage

The objectives of agricultural drainage are:

(i) The orderly removal of water that is excess to economic crop production.

(ii) The establishment and maintenance of a proper salt balance in the soil, since under certain conditions, and especially in arid countries, inadequate drainage can cause excessive concentrations of salt.

The importance of drainage is indicated by the fact that it is believed that inadequate drainage contributed significantly to the decline of many early civilizations and is a threat to many present irrigation schemes (Hagan, Haise, and Edminster [1967, p. 988]).

Basic references on drainage include Luthin [1957], van Schilfgaarde [1970, 1974].

Drainage FBPS consist of combinations of the following components:

(a) Sources: Surface water applied either by rainfall or sprinkler; artesian water rising from underground; canals; "foreign water" flowing from a catchment area.

(b) Sinks: Open ditches; mole drains (unlined); tile drains (lined). In the case of drains it is usual to treat the drains as line sinks; however, drains of finite cross-section are sometimes considered. Tile drains are lined with a porous material which introduces an additional resistance to flow (usually neglected).

Geometry: The two basic geometries are shown in Figures 1 and 2.

In Figure 1 a series of parallel drains is fed by a uniform rainfall. In the case of line sinks, infinite depth, with or without a capillary fringe an analytical solution can be obtained by the hodograph method (see Childs [1959], van Schilfgaarde [1970, p. 65]). The case of line sinks in soil of uniform depth (as in Figure 1) has been solved approximately using infinite series by Kirkham [1966], and List [1964] gives another approximate solution.

Childs has used the trial-free-boundary method with conducting paper to study several parallel drain problems for soil of uniform depth: Childs [1943] considers circular drains of finite diameter; Childs [1945] considers (a) the effect of slits made to observe the water table, (b) the relative efficacy of flooded and empty drains, (c) the effect of putting a drain at the bottom of a ditch and filling in, (d) the role of the capillary fringe; Childs [1945a] considers the validity of the assumption that the flow of rainwater is uniform across the FB. Van Deemter [1950] solved several parallel drain problems using the trial free boundary method with finite differences.

Figure 2 shows a drain across a sloping bed down which "foreign water" flows at a steady rate. Childs [1946] uses the trial-free-boundary method with conducting paper to study this problem for both tile drains and open ditches.

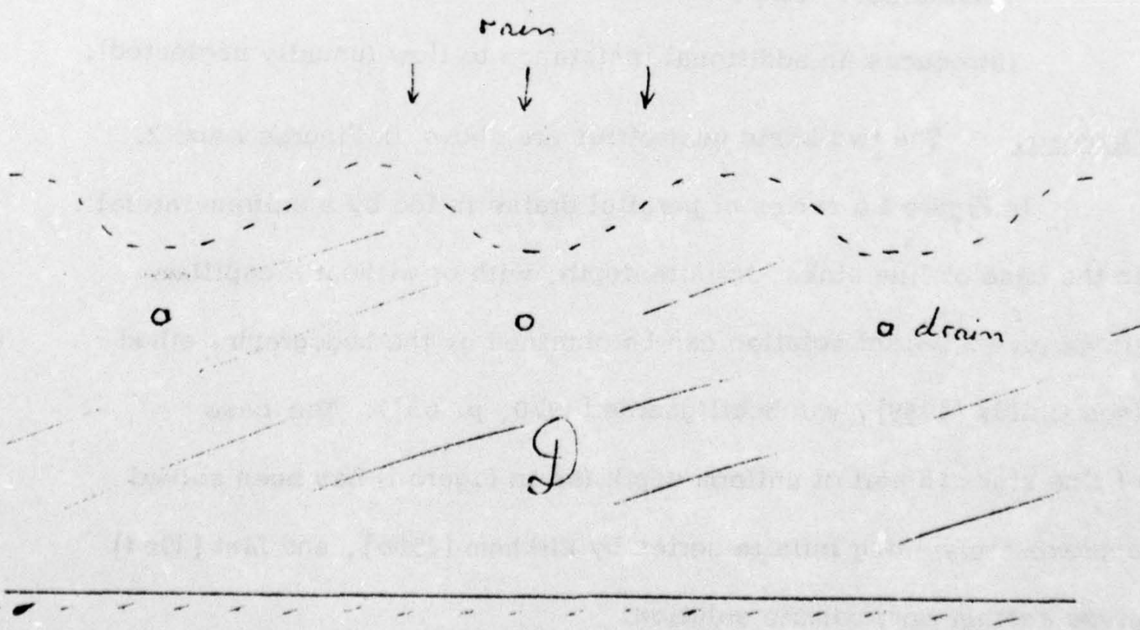


Figure 1: Parallel drains in soil of uniform depth

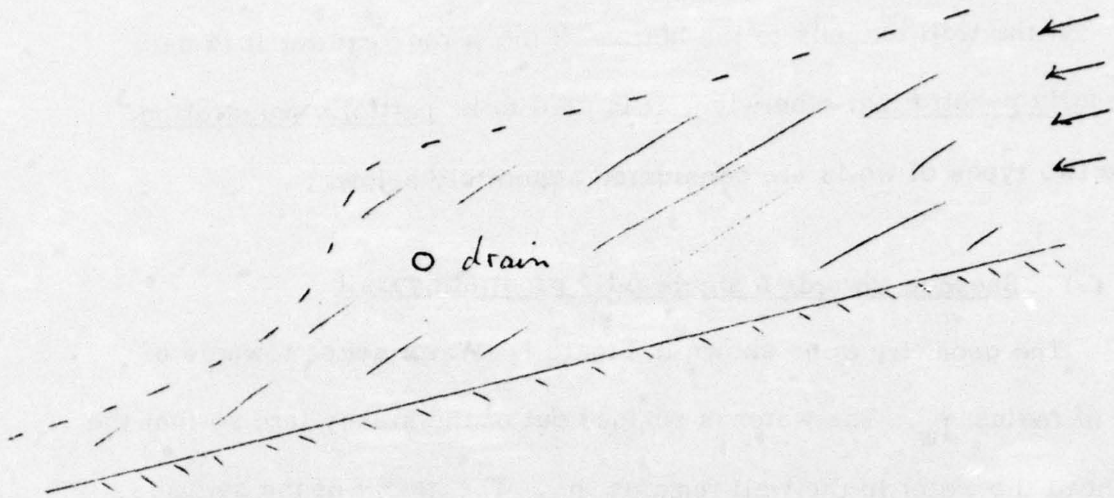


Figure 2: Foreign water draining down a slope

We conclude with some remarks:

(i) Drainage problems involving drainage into ditches are equivalent to dam problems (see section 3.1). Problems involving ditches are also considered in section 3.2.

(ii) Drainage FBPS have been relatively little studied in comparison with problems such as porous dam FBPS, and there is scope for further work such as the application of variational inequalities.

3.4. Seepage towards wells or ditches

Wells play an important role in the use of water. The books of Harr [1962, chapter 10] and Polubarinova-Kochina [1962, chapter 9] provide much information, and Hantush [1964] gives a lengthy survey.

Wells also play an important role in the recovery of oil and gas, but these problems involve two fluids and are discussed in section 4.

3.4.1. Seepage towards a single well

If the well extends to the bottom of the porous aquifer it is said to be fully penetrating; otherwise, it is said to be partially penetrating. These two types of wells are considered separately below.

3.4.1.1. Seepage towards a single fully penetrating well

The geometry is as shown in Figure 1. Water seeps towards a well of radius r_w. The water is pumped out at the steady rate so that the height of the water in the well remains h_w. The height of the seepage surface is h_s. The catchment area of the well is of radius r_e and height h_e. The quantity $h_e - h_s$ is called the draw-down.

The boundary conditions on two of the boundaries require comment:

(i) The water drawn from the well must be balanced by a flow into the region at the outer boundary $r = r_e$. It is usual to assume that $h = h_e$ at $r = r_e$; this is equivalent to assuming that the catchment area is surrounded by water so that the well is the axially symmetric equivalent of flow through a rectangular dam.

(ii) At the well face $r = r_w$ it is usually assumed that $h = h_w$ if $y \leq h_w$ and $h = y$ if $y \geq h_w$. However, wells are often lined by a porous material which reduces the flow, and Hall [1955, p. 30] allows for this by setting $h = y + \bar{h}_{max}$ on the seepage face where $\bar{h}_{max}$ is a positive constant.

The subsections below consider the homogeneous and inhomogeneous cases, respectively.

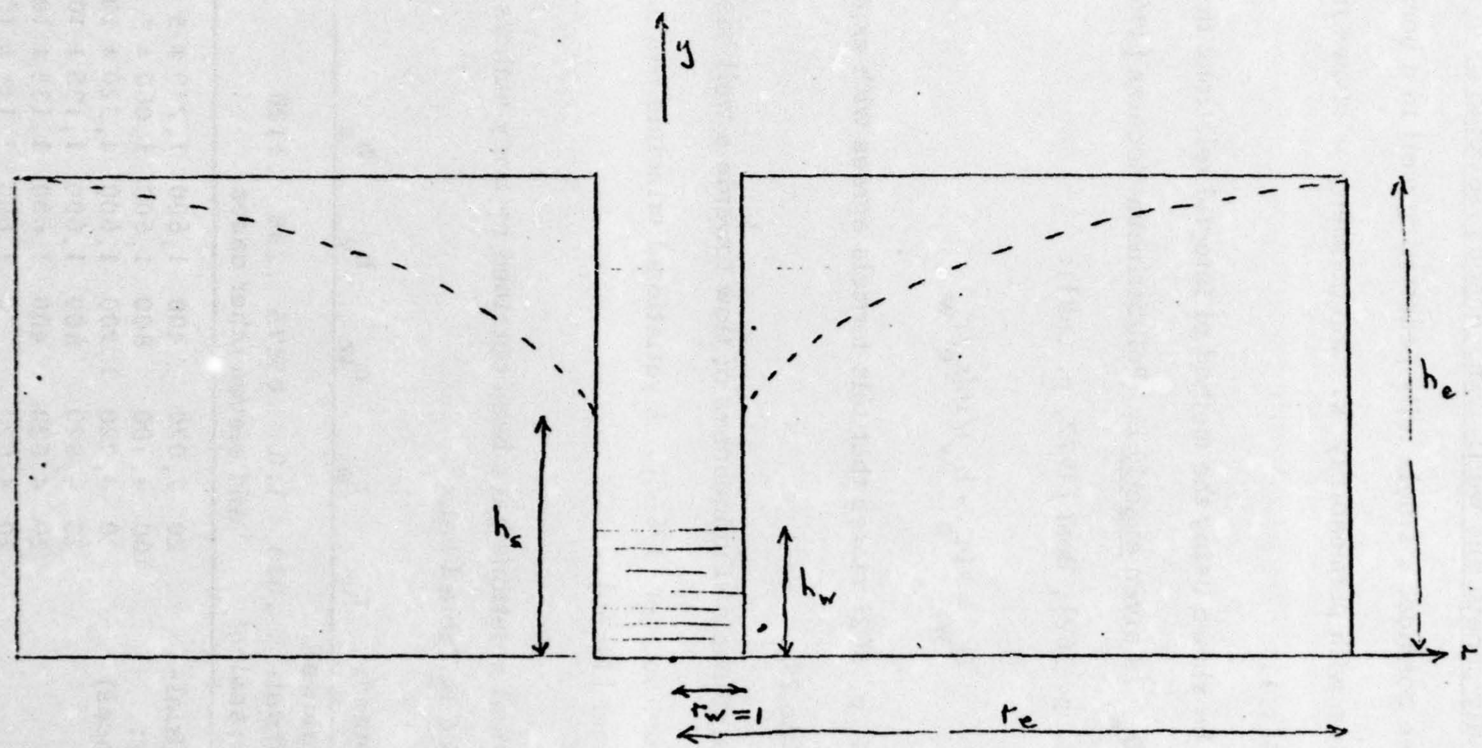


Figure 1: Seepage towards a single fully-penetrating well

3.4.1.1.1. Fully penetrating well: homogeneous isotropic case

Here we consider a single fully penetrating well in a homogeneous isotropic medium with permeability k. The geometry is shown in Figure 1 of section 3.4.1.1.

It can be shown using the method of integral relations that the rate of flow Q_w is given exactly by (Polubarinova-Kochina [1962, p. 283], Hantush [1964, p. 362], Bear [1972, p. 368]):

$$Q_w = k(h_e - h_w)/\ln(r_e/r_w) \; . \tag{1}$$

Hantush [1964, p. 362] states that this formula agrees with experimental results to within 2%.

The existence and uniqueness of flow towards a well has been studied by Mauersberger [1965c]. A variational principle has been derived by Mauersberger [1965b].

Numerical solutions have been obtained by many authors and are summarized in Table 1 below.

Author (Method)	r_w	r_e	h_w	h_e	h_s
Babbitt and Caldwell [1948, p. 27] (trial-free-boundary; analog)	.083	1.0	.05875	.235	.1128
		and eleven other cases			
Yang [1949] (trial-free-boundary; finite differences)	20	2,020	400	1,600	1,240 ± 5
	100	4,100	800	1,600	1,060 ± 5
	20	4,020	1,200	1,600	1,320 ± 10
	20	4,020	800	1,600	1,195 ± 10
	20	4,020	400	1,600	1,135 ± 10
	20	4,020	0	1,600	1,115 ± 10

Author (Method)	r_w	r_e	h_w	h_e	h_s
Boulton [1951] and J. A. Murray [1960] (trial-free-boundary; finite differences)	50	6,400	3,200	6,760	5,725
	200	10,400	3,200	7,165	5,475
	200	10,400	0	7,120	5,153
	1,000	16,000	0	4,000	1,500
Kashef, Touloukhan, and Fadum [1952, p. 73] (*trial free boundary; finite differences*). See also Kashef [1953].	100	4,100	800	1,600	1,040*
H. Schmidt [1956] (time-dependent; finite differences)	1,000	30,000	10,000	8,000	8,152
	1,000	30,000	10,000	6,000	6,695
	1,000	30,000	10,000	4,000	5,711
	1,000	30,000	10,000	2,000	5,168
	1,000	30,000	10,000	0	4,894
H. P. Hall [1955] (trial-free-boundary; finite differences)	4.8	76.8	36	48	Graphical (see remarks)
	4.8	76.8	24	48	
	4.8	76.8	12	48	
Herbert [1968, p. 260] (time-dependent; resistance network)	11	99	35	70	54.5
G. S. Taylor and Luthin [1969] (time-dependent; finite differences)	4.8	76.8	12	48	Graphical
Neuman and Witherspoon [1970] (trial-free-boundary; finite elements)	4.8	76.8	12	48	Graphical (see remarks)
Neuman and Witherspoon [1971a] (time-dependent; finite elements)	4.8	76.8	12	48	Graphical
France, Parekh, Peters, and Taylor [1971, p. 176] (time-dependent; finite elements)	11	99	35	70	46*

*Estimated by us from a graph

Table 1: Axisymmetric flow towards a fully penetrating well

As can be seen from Table 1, a large number of different cases have been considered, so that it is difficult to compare the results. It is also unfortunate that some of the results are given in graphical form. However, the case $r_w = 4.8$, $r_e = 76.8$, $h_w = 12$, $h_e = 48$ has been considered by Hall [1955], G. S. Taylor and Luthin [1969], and Neuman and Witherspoon [1970], and Neuman and Witherspoon [1970] give a graphical comparison of these results. The results are not strictly comparable, because Hall assumed capillarity ($h_c = 3.6$) and a lined well ($\bar{h}_{max} = 1.0$), Taylor and Luthin assume partially saturated flow, and Neuman and Witherspoon assume saturated flow without capillarity, but the results are quite close. France, Parekh, Peters, and Taylor [1971] give a graphical comparison of their results with those of Herbert [1968].

R. L. Taylor and Brown [1967] give graphical results which were obtained using a general program. This program is listed by R. L. Taylor [1966], and an improved version is listed by Kealy and Busch [1971].

There are several approximate analytic solutions in the literature. In a series of papers, Mauersberger has obtained approximate solutions using the variational method of Trefftz (Mauersberger [1967, 1968, 1968a]), collocation (Mauersberger [1968, 1968b, 1968c]), least squares (Mauersberger [1968, 1968c]), Galerkin's method (Mauersberger [1968, 1968c]), the "partition method" (Mauersberger [1968, 1968c]). Kirkham [1964] has obtained an approximate solution by introducing a "fictitious flow region" and using collocation. Kirkham does not give specific numbers and checks his solution by comparing it with the problem of flow through a dam.

Brutsaert, Breitenbach, and Sunada [1971] solve numerically the corresponding time-dependent multiphase problem.

3.4.1.1.2. Fully penetrating well: the general case

The case which has been studied the most intensively is the case of N parallel aquifers (Figure 1): aquifer i is of depth a_i and has permeability k_i. (Note that the ordering of the aquifers is sometimes reversed, aquifer 1 being at the bottom.) In other words,

$$k = k(y) = k_i, \quad \text{if} \quad \sum_{j=i-1}^{N} a_j \leq y \leq \sum_{j=i}^{N} a_j . \tag{1}$$

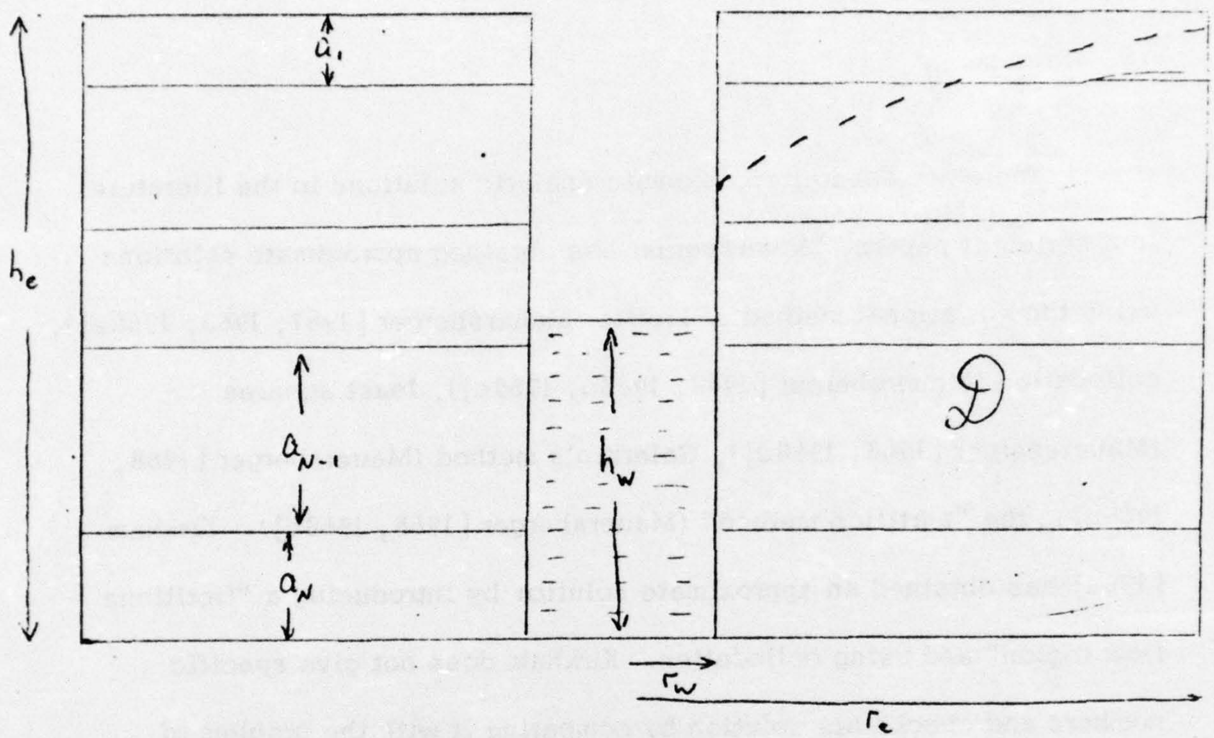


Figure 1. A single fully-penetrating well in a multiple aquifer

Neumann and Witherspoon [1969] give a detailed discussion of transient flow for a multiple aquifer (Figure 1) under the assumption that the FB is horizontal.

Mauersberger [1969] shows that it is possible to use the method of integral relations to obtain an expression for the rate of flow Q_w when the permeability k is of the form

$$k = k(x, \theta, y) = k_1(x)k_2(\theta, y)$$

where (x, θ, y) are axial cylindrical coordinates. This includes the cases

$k = k_1(x)$ and $k = k_2(y)$ as special cases. In particular, Mauersberger gives the value of Q_w when k is of the form (1).

Youngs [1971b] considers the case when $k = k(y)$ and when precipitation occurs at a rate R per unit area. For the special case

$$k(y) = \begin{cases} k_2, & 0 < z < a_2, \\ k_1, & a_2 \leq z, \end{cases}$$

$h_w = a_1 + \bar{a}_2$, and $h_e = a_1 + \bar{a}_2 + H'$, Youngs obtains the formula

$$\frac{Q_w}{\pi r_w^2} = \frac{(H')^2 k_1 R}{R r_w^2 \ell n(r_e/r_w)} + \frac{2RH'(k_1 a_1 + k_2 \bar{a}_2)}{R r_w^2 \ell n(r_e/r_w)} + \frac{R[(r_e/r_w)^2 - 1]}{2\ell n(r_e/r_w)} - R .$$

If there is no precipitation $(R = 0)$ then this formula of Youngs is of course contained in the results of Mauersberger.

Lewis and Humpheson [1973] use finite elements to solve the time-dependent problem of a well subjected to electrical forces (Figure 2).

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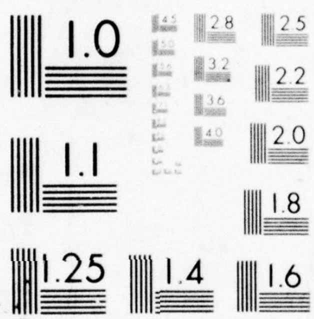

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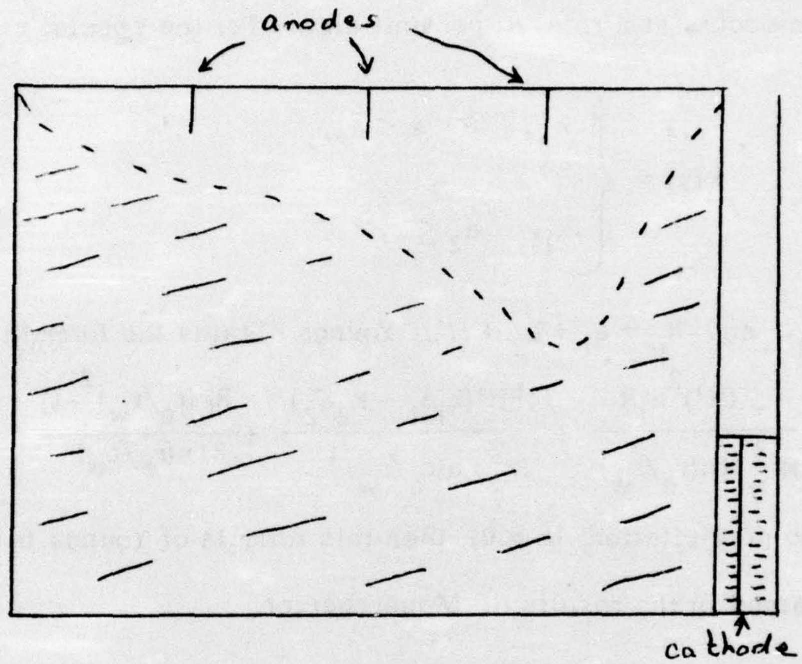


Figure 2. Free surface profiles under varying voltage gradients after 2 1/2 days (based on Lewis and Humpheson [1973, p. 612])

3.4.1.2. Seepage towards a single partially penetrating well

The geometry is shown in Figure 1.

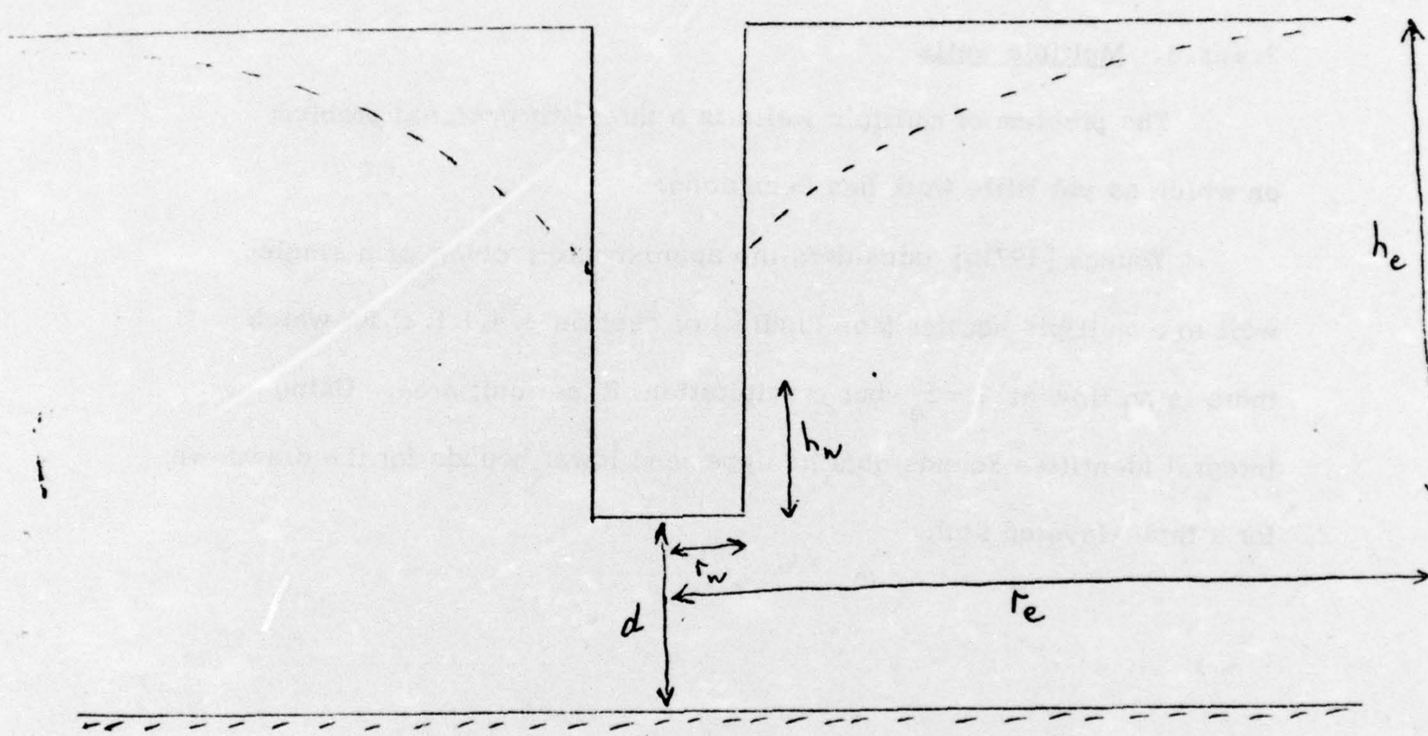


Figure 1. A single partially penetrating well

Numerical solutions have been obtained by several authors: Boreli [1955] (trial-free-boundary; finite differences); R. L. Taylor and Brown [1967] using the program of R. L. Taylor [1966] (trial-free-boundary; finite elements); Cooley [1971] (time-dependent partially-saturated flow; finite differences). Unfortunately, only Boreli states the dimensions clearly, and all the authors give their results in graphical form.

Breitenöder [1942, p. 84 and p. 86] considers the plane problem for $d = r_e = \infty$.

It would seem that the problem deserves further study.

3.4.1.3. Multiple wells

The problem of multiple wells is a three-dimensional problem on which as yet little work has been done.

Youngs [1971b] considers the approximate problem of a single well in a multiple aquifer (see Figure 1 of section 3.4.1.1.2) for which there is no flow at $r = r_e$ but precipitation R per unit area. Using integral identities Youngs obtains upper and lower bounds for the drawdown for a three-layered soil.

4. Two-fluid flows

Two-fluid porous flow FBS arise when a lighter fluid (fluid 1) lies on top of a heavier fluid (fluid 2), the interface between the fluids being a FB. In the analysis of such problems it is usually assumed that the lower fluid is at rest. The boundary conditions on the interface are given in section 2.2.

Two-fluid porous flow FBPS arise in two important contexts namely sea water/fresh water FBPS in coastal areas and water/gas/oil FBPS in the oil industry, and these are discussed in the first two subsections below. The final subsection discusses the FBPS which arise when one considers the FB at the microscopic level.

4.1. Salt water - fresh water interfaces

In coastal regions there is an interaction between fresh water being applied to the ground surface by rainfall or irrigation, and salt water from the sea. The fresh water is lighter and forms a layer on top of the salt water. It is of importance to prevent the encroachment of the sea water because the salt destroys agricultural land and contaminates water supplies.

In studying salt water/fresh water problems it is usual to assume that the salt water is at rest, and this assumption will be made throughout unless explicitly stated otherwise. Indeed, sc far as we are aware only Childs [1950] has considered the case when the salt water is in motion. In this connection it is of interest that the observed fresh water/salt water interface in the Miami

area shows that motion of the sea water must occur (Cooper, Kohout, Henry, and Glover [1964, p. C12]).

Of course, the assumption of a fresh water/salt water interface is only an approximation, and in the general case it is necessary to take account of the diffusion of the salt.

4.1.1. Salt water - fresh water: the Ghyben-Herzberg lens

The first salt water - fresh water FBP to be studied occurs when fresh water is provided by either rainfall or irrigation to an island (axisymmetric) or isthmus (plane). A bubble of fresh water, often called the Ghyben-Herzberg lens, forms underneath the island (Figure 1). There are two FBS: the fresh water/air interface and the fresh water/salt water interface.

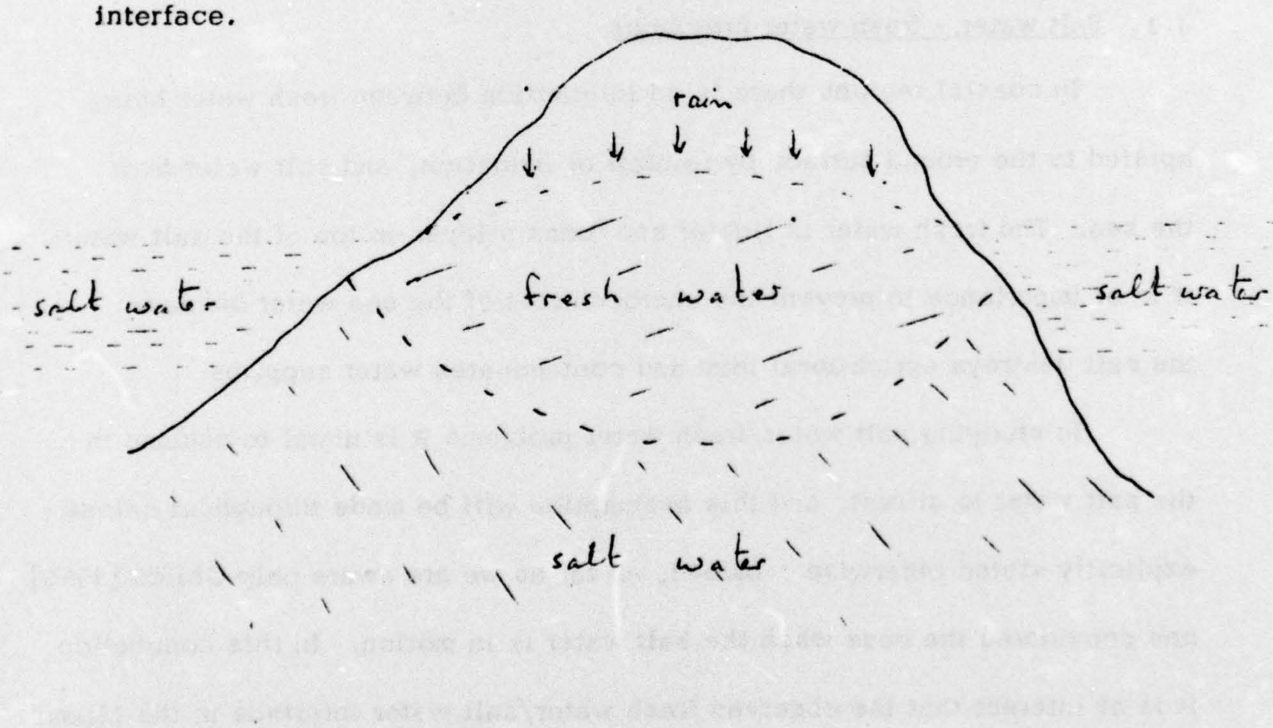


Figure 1. A Ghyben-Herzberg lens

Childs [1950] used the trial-free-boundary method with conducting paper to obtain the solution for the case of a rectangular isthmus. Childs allows the salt water to be at different levels, which can arise because of tidal effects (Figure 2). Youngs [1971, 1971a] obtains bounds for the rate of flow with and without a well on the island.

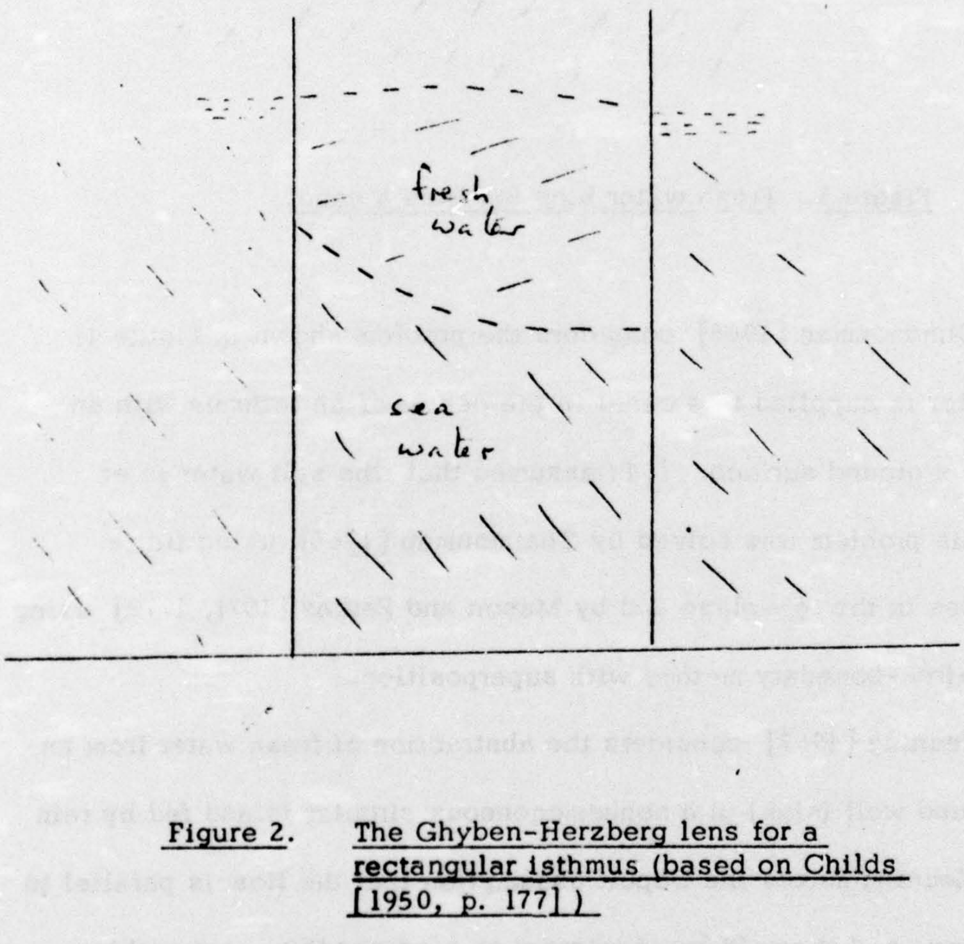


Figure 2. The Ghyben-Herzberg lens for a rectangular isthmus (based on Childs [1950, p. 177])

Ackerman and Intong [1966] consider a related problem in which fresh water is supplied from a canal with impervious walls. It is assumed that the upper surface is known and that the salt water is at rest (Figure 3). An analytical solution is found using the hodograph method.

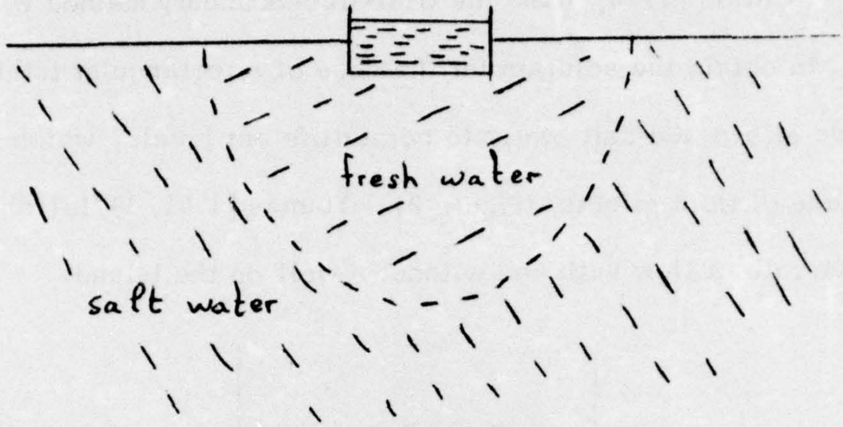


Figure 3. Fresh water lens fed from a canal

Charmonman [1966] considers the problem shown in Figure 4: fresh water is supplied to a canal in the center of an isthmus with an impervious ground surface. It is assumed that the salt water is at rest. This problem was solved by Charmonman [1966] using finite differences in the $\phi\psi$-plane and by Mason and Farkas [1971, 1972] using the trial-free-boundary method with superposition.

Keuning [1967] considers the abstraction of fresh water from an underground well (sink) in a nonhomogeneous circular island fed by rain water. Keuning makes the Dupuit assumption that the flow is parallel to the surface, and it would be of interest to consider the exact problem.

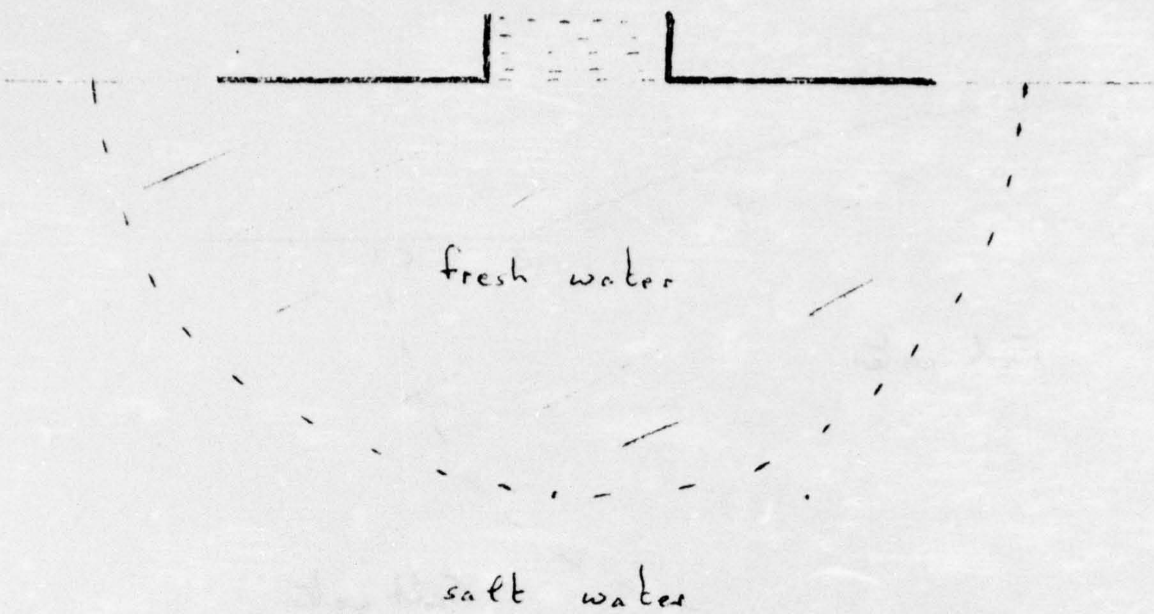


Figure 4. Canal with impervious surface

4.1.2. Coastal aquifers

The geometry is shown in Figure 1: fresh water flows from a coastal aquifer into the sea. There are two FBS namely the upper fresh water/air FB and the lower fresh water/salt water FB. Charmonman [1965] uses the hodograph method to obtain an analytical solution for the case when the outflow surface BC is horizontal.

Henry [1959] uses the hodograph method to obtain the exact solutions for flow from a horizontal aquifer of finite depth with vertical outflow face (Figure 2a) and horizontal outflow face (Figure 2b). (The problem of Figure 2b for the case of infinite depth was solved earlier by Glover [1959]).

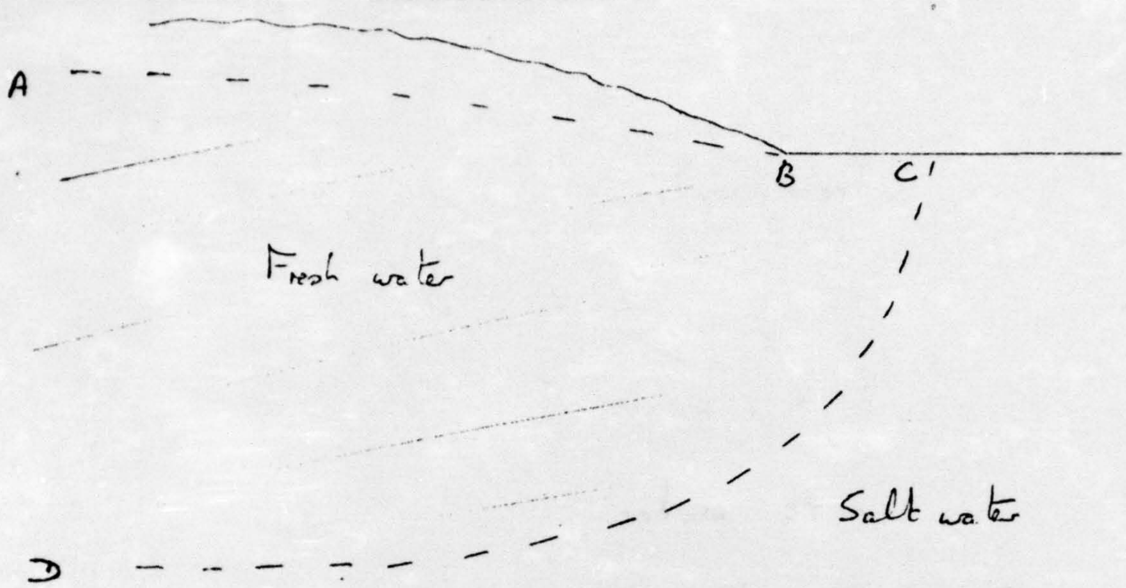
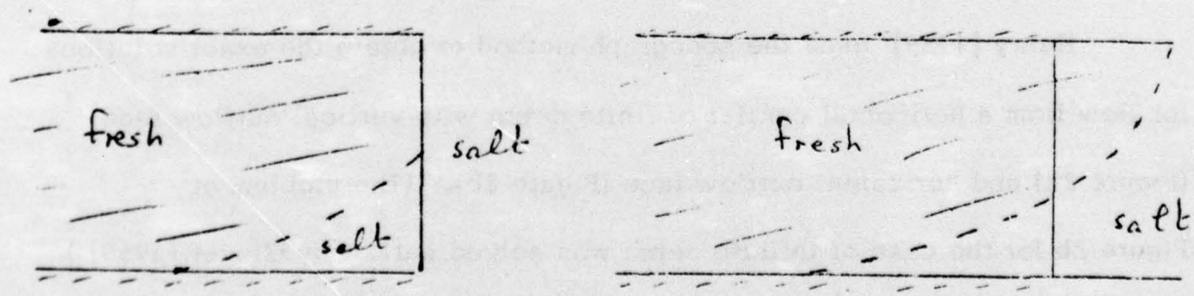


Figure 1. Flow from a coastal aquifer

The coupled equations for Darcy flow of water and dispersion of salt have been solved numerically for the problem of Figure 2a by Henry (see Cooper, Kohout, Henry, and Glover [1964, p. C70]) and Pinder and Cooper [1970] .



(a) vertical outflow (b) horizontal outflow

Figure 2. Horizontal aquifers of finite depth

De Jong [1965] uses the hodograph method to obtain the analytical solution for flow in a semi-infinite aquifer with a surface sink (Figure 3). De Jong also gives experimental results which are in excellent agreement with the theoretical results.

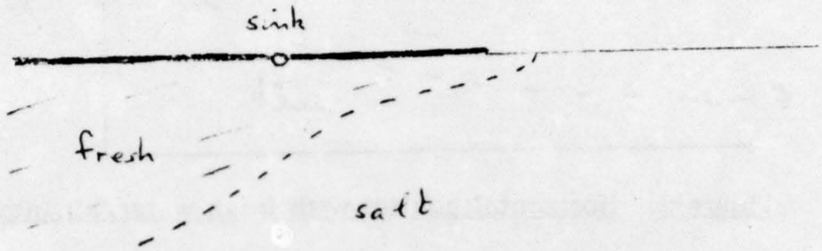


Figure 3. Semi-infinite aquifer with sink

Baiocchi, Comincioli, Magenes, and Pozzi [1973, p. 8 and p. 39] consider the generalization of the problem of Figure 2a obtained by allowing a fresh water/air interface (Figure 4). It is assumed that the aquifer is so wide that FG is a line of equipotential. Baiocchi, Comincioli, Magenes, and Pozzi reformulate the problem as a variational inequality and prove existence and uniqueness of the solution. Baiocchi, Comincioli, Guerri, and Volpi [1973, p. 29] use variational inequalities with finite differences to obtain a numerical solution. Youngs [1971a] derives the exact value of the rate of flow.

In some coastal aquifers the aquifer is divided into several strata by interlying semi-pervious layers. As a result, seawater intrudes into each of the separate aquifers and forms a series of wedges. Such coastal aquifers occur near New York, and off the coasts of Israel and the Netherlands (M. A. Collins and Gelhar [1971]). The geometry is

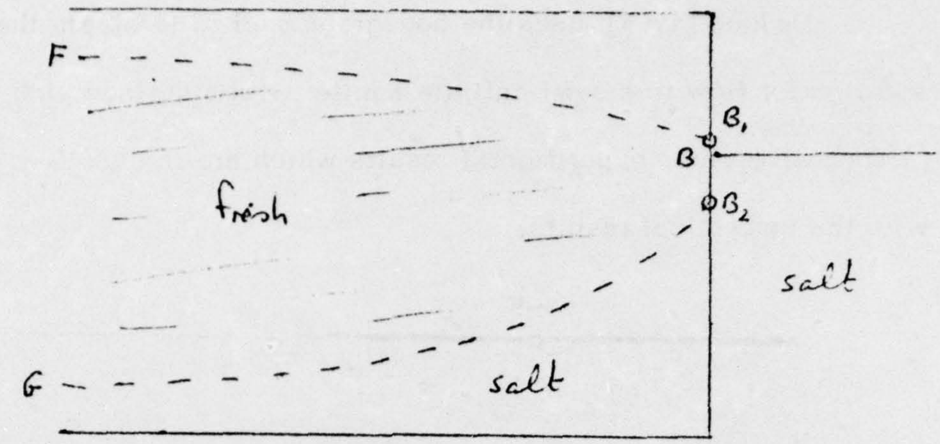


Figure 4. Horizontal aquifer with fresh water/air interface

shown in Figure 5. Collins and Gelhar [1971] give an approximate analysis based upon the Dupuit approximation.

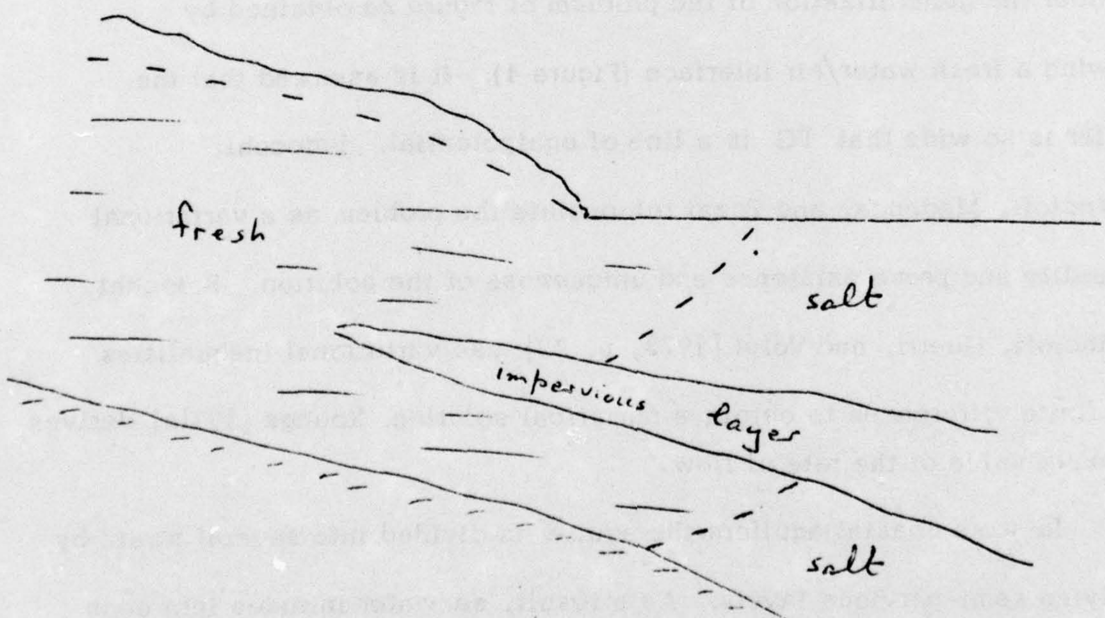
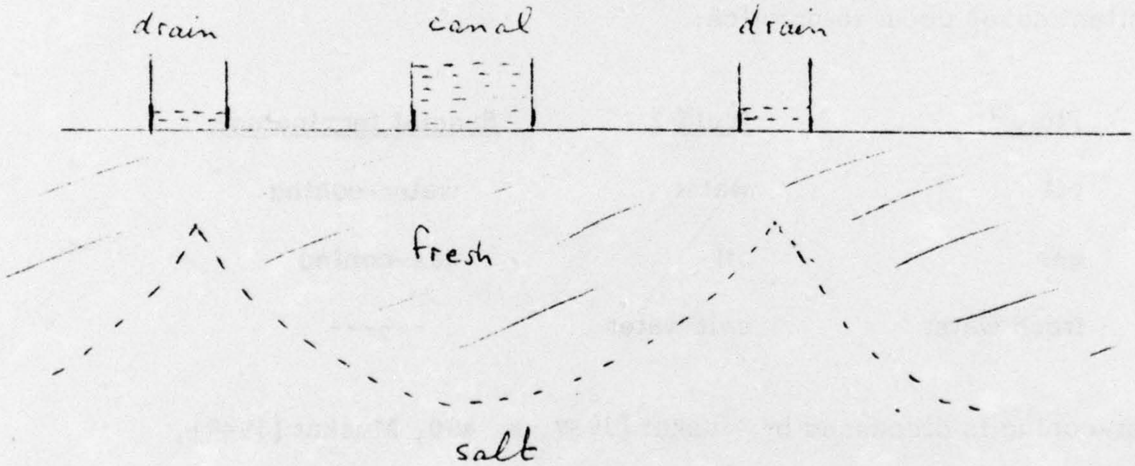


Figure 5. A layered aquifer

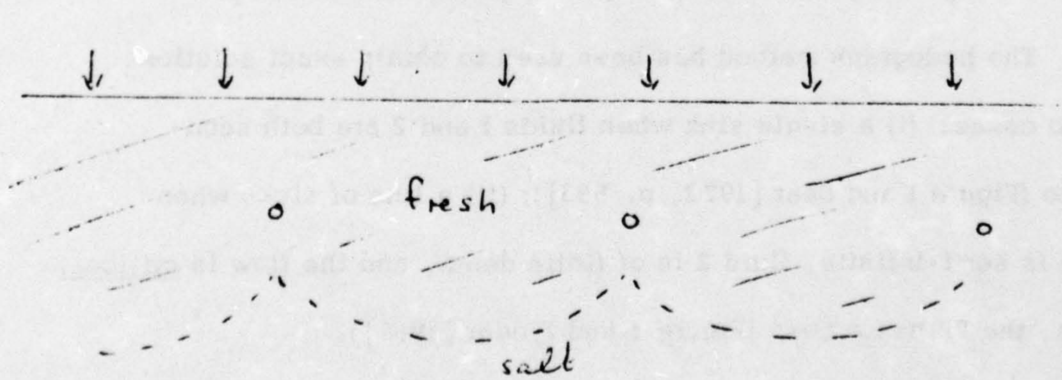
4.1.3. Land reclamation

In certain areas of the world, such as the central plain of Thailand, it is conceivable that it will be possible to reclaim land from the sea by applying fresh water to the surface.

Charmonman [1967] uses finite differences in the $\phi\psi$-plane to analyse an array of parallel drains and canals (Figure 1a). Ackerman and Chang [1971] use the hodograph method to obtain an analytical solution for the case when fresh water is applied to the surface and then (to reduce the loss of fresh water) pumped from a series of drains (Figure 1b).



(a) Periodic canals and drains



(b) Periodic drains with surface application

Figure 1. Methods of land reclamation

4.2. Other configurations

4.2.1. Up-coning

When a layer of one fluid (fluid 1) lies over a layer of a second fluid (fluid 2) and fluid 1 is pumped out from one or more wells (or sinks) then the interface between the fluids moves towards the wells and this is called up-coning because the interface often approximately assumes the shape of a cone. If the rate of pumping is too high then fluid 2 may enter the well and contaminate the well output. Three especially important cases occur in practice:

Fluid 1	Fluid 2	Special terminology
oil	water	water-coning
gas	oil	gas-coning
fresh water	salt water	------

Water-coning is discussed by Muskat [1937, p. 480, Muskat [1949], Pirson [1958]; gas-coning is discussed by Muskat [1937, p. 689]; and up-coning is discussed by Bear [1972, p. 553 and p. 569].

The hodograph method has been used to obtain exact solutions for two cases: (i) a single sink when fluids 1 and 2 are both semi-infinite (Figure 1 and Bear [1972, p. 553]); (ii) a line of sinks when fluid 1 is semi-infinite, fluid 2 is of finite depth, and the flow is critical, that is, the FB has a cusp (Figure 2 and Kidder [1956]).

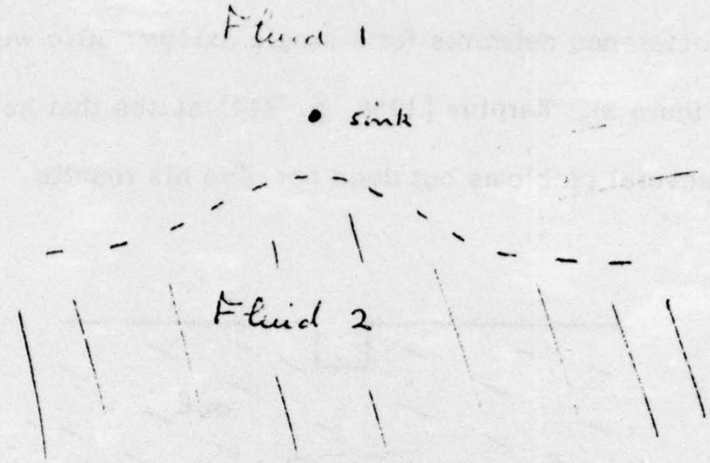


Figure 1: A single sink and two semi-infinite fluids

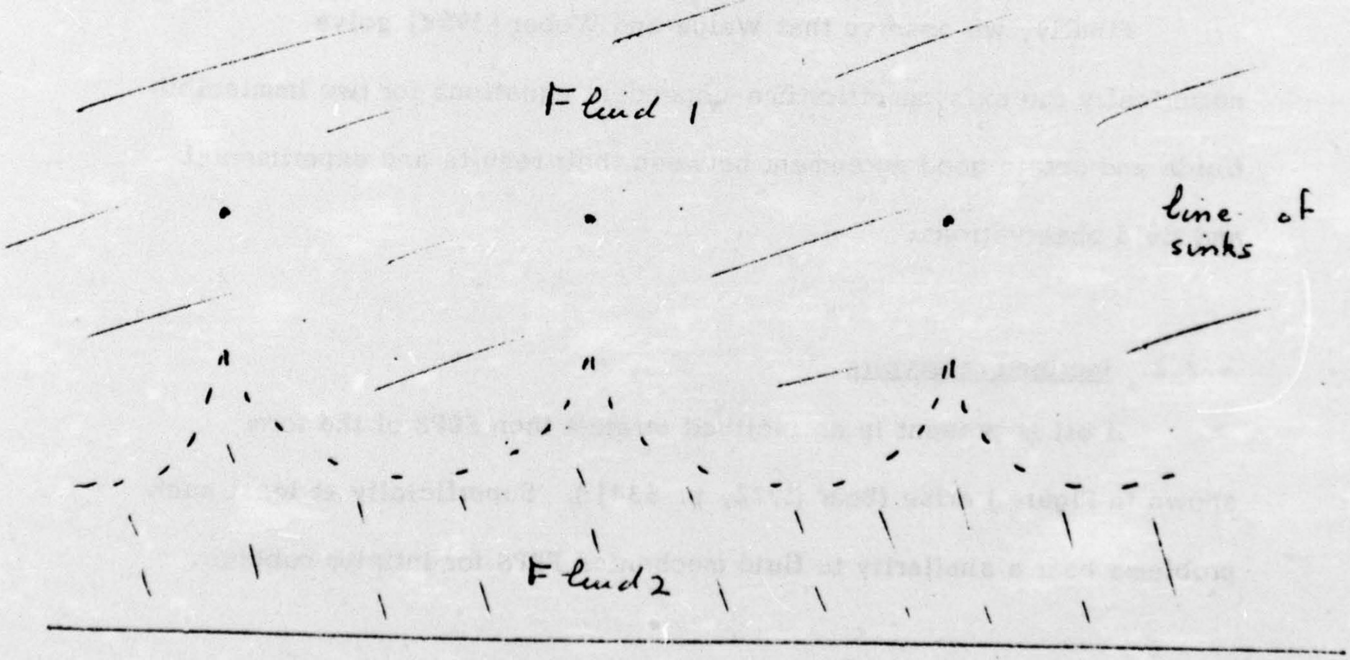


Figure 2: Critical flow for a line of sinks with fluid 1 semi-infinite and fluid 2 bounded

Karplus [1956] describes the use of the trial-free-boundary method with resistance networks for a single axisymmetric well of finite dimensions (Figure 3). Karplus [1956, p. 244] states that he has investigated several problems but does not give his results.

Figure 3: Water cone in vicinity of oil well
(based on Karplus [1956, p. 241])

Finally, we observe that Welge and Weber [1964] solve numerically the axisymmetric time-dependent equations for two immiscible fluids and obtain good agreement between their results and experimental and field observations.

4.2.2. Inclined reservoirs

If oil is present in an inclined stratum then FBPS of the form shown in Figure 1 arise (Bear (1972, p. 534]). Superficially at least such problems bear a similarity to fluid mechanics FBPS for infinite bubbles.

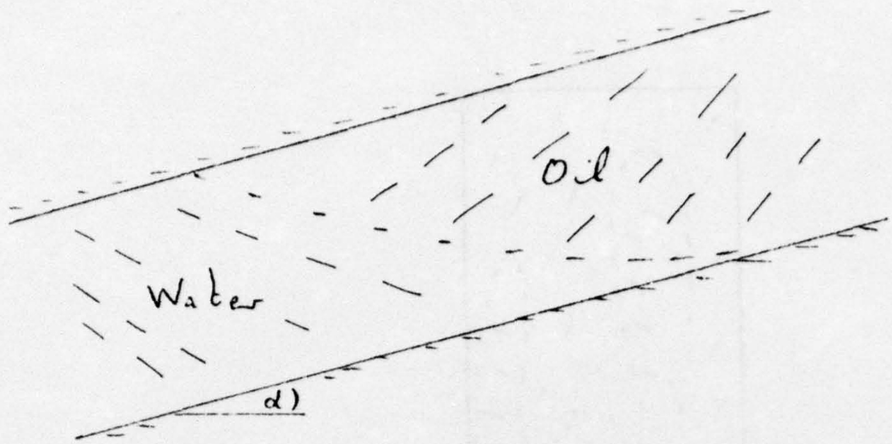


Figure 1. An inclined plane reservoir

4.3. Interfacial instability

When a viscous fluid 1 in a porous medium is driven forwards by the pressure of another driving fluid 2, the interface between the two fluids is liable to be unstable if $\mu_2 < \mu_1$ where μ_1, μ_2 are the viscosities of the two fluids. Heuristically, one can see why such instability occurs: the driving fluid (fluid 2) encounters less resistance than fluid 1 so that for a given driving pressure the flow rate will be greater if the interface is distorted so that fluid 2 can flow leaving some of fluid 1 behind.

Saffman and Taylor [1958] modeled the problem using the analogy between porous flow and viscous flow in a Hele-Shaw cell (see Bear [1972, p. 687]). Figure 1 shows how an interface, which was originally straight, deforms into a number of fingers, and for this reason the phenomen is called fingering.

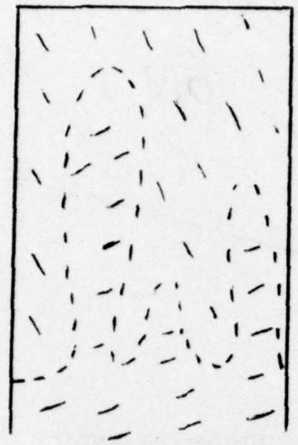


Figure 1: Fingering (based on Saffman and Taylor [1958, p. 319])

The phenomenon of fingering can occur in oil production in which a more viscous fluid (oil) is being driven by a less viscous fluid (water). It seems to us, however, that there is some confusion in the literature between fingering and water-coning. Figure 2 shows a line of oil wells producing oil.

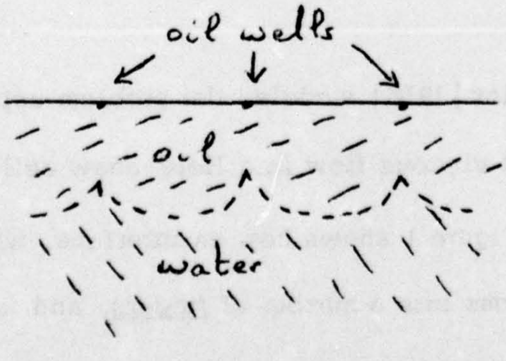


Figure 2: Water-coning (based on Kidder [1956, p. 867])

The oil/water interface is distorted in the neighborhood of each well, and "fingers" are formed. However, the mechanism of the phenomena showed in Figures 1 and 2 is quite different: in Figure 1 the driving pressure is uniform and the fingers are caused by the instability of the surface; in Figure 2 the "fingers" correspond to the spacing of the wells. It thus seems to us that it is incorrect to compare fingering and water-coning as done by Saffman and Taylor [1958, p. 314]. Water-coning is discussed in section 4.2.

To analyse fingering, Saffman and Taylor [1958, p. 318] consider the case when there is an infinite set of equal and equally spaced fingers all advancing at the same speed. Since each finger is then identical mathematically with all the others and the fluid on the straight lines halfway between neighbors has no transverse component of velocity, it suffices to consider a single finger propagating itself in a channel of fixed width.

Saffman and Taylor use the hodograph method to obtain an analytical solution, or rather a family of solutions, because the solution contains the parameter

$$\lambda = \frac{\text{width of channel}}{\text{asymptotic width of finger}} .$$

Experimentally, Saffman and Taylor find that λ is very close to 1/2, but they are unable to explain this: the maximum velocity occurs when $\lambda = 0$; the maximum and minimum rate of energy dissipation occur

when $\lambda = 0$ or $\lambda = 1$; a stability analysis shows that the motion is unstable for all λ.

Saffman [1959] obtains a family of exact solutions of the time-dependent problem which begin as an approximately sinusoidal perturbation and grow into fingers. Saffman [1959, p. 159] observes that the growth is slowest when $\lambda = 1/2$ but adds that the physical significance of this is not clear. Similar exact solutions were obtained by Jacquard and Seguier [1962].

We conclude with some observations:

(i) Bear [1972, p. 544] discussed fingering and gives references. The approach followed is different from that of Saffman and Taylor, and emphasizes the conditions for stability or instability rather than the shape of the fingers.

(ii) There is considerable similarity between fingering and other interfacial instability phenomena such as:

(a) Dendrite formation in crystals.

(b) Lubrication cavitation in bearings.

(c) Taylor instability at the accelerated interface between fluids of different densities.

(iii) As Saffman and Taylor observe, the non-uniqueness of fingers (because of the arbitrary parameter λ) is similar to the non-uniqueness

of bubbles moving in tubes. In both cases it would seem that an additional physical condition must be imposed.

(iv) S. Richardson [1972] considers similar problems arising in the injection molding of plastics.

(v) The analysis of Saffman and Taylor is for plane problems; so far as we are aware, axisymmetric problems have not been considered.

5. Coupled-field problems

Under coupled-field porous flow FBPS we understand porous flow FBPS which involve, in a non-trivial way, one of the other fields of continuum mechanics such as heat flow on electromagnetism.

There are many possible coupled-field FBPS involving dispersion or diffusion. Bear [1972, chapter 10] gives an excellent discussion and lists the following applications: (a) the transition zone between salt water and fresh water in coastal aquifers; (b) artificial recharge operations where water of one quality is introduced into aquifers containing water of a different quality (mixing of water due to hydrodynamic dispersion is among the various objectives of artificial replenishment); (c) secondary recovery techniques in oil reservoirs, where an injected fluid dissolves the reservoir's oil; (d) radioactive and reclaimed sewage waste disposal into aquifers; (e) the use of tracers, such as dyes, electrolytes and radioactive isotopes, in hydrology, petroleum engineering and other scientific and engineering research projects; (f) the use of reactors packed with granular material in the chemical industry; and (g) the movement of fertilizers in the soil and the leaching of salts from the soil in agriculture. We have not searched the literature for FBPS involving these phenomena, although such problems doubtless occur.

Another important coupled-field problem involves heat transfer. Kirkham and Powers [1972, p. 462] observe that soil temperature affects the germination of seeds, the chemical reactions which release nutrients,

and the availability of water. Also (Kirkham and Powers [1972, p. 483]) the freezing of soil is of importance in agriculture and in civil engineering projects involving the building of roads and pipelines. Bear [1972, p. 641] discusses heat transfer in a porous medium. Here again, we have not searched the literature for FBPS.

A third coupled-field problem arises in connection with electrokinetic effects: the movement of water under the influence of an electric field. This is duscussed further in section 1.6 and the flow towards a well has been treated by Lewis and Humpheson [1973] (see section 3.4.1.1.2).

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Each item in the bibliography is followed by a list of the sections in which it is quoted. The following abbreviations are used:

DA: Dissertation Abstracts

MR: Mathematical Reviews.

If an author has more than one publication in year X, his publications are numbered X, Xa, Xb, ... This bibliography is part of a larger bibliography on FBPS, and to maintain compatability we use the same numbering, so that if references X and Xa are not concerned with porous flow FBPS we only include reference Xb but retain the number Xb.

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A comprehensive survey of steady-state, porous flow, free boundary (unconfined flow) problems is given. The problems are described and the numerical and analytical approaches which have been used are summarized. Attention is drawn to unsolved problems, and open questions.

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